Lizardy Loops: The Savvy Selection of Sinuous Sequences of Circular Sectors

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Abstract

This paper attempts to answer a simple question: how can one combine sequences of circular sectors of angle $2\pi/n$ to form closed loops? A number of methods for systematically creating symmetric and non-symmetric loops of such sectors will be explored. These sequences are beautiful in their own right, and can also be used as frameworks for the creation of more elaborate art pieces.

Defining the Problem

Consider a circle divided into n circular sectors. These sectors can be connected along their radial edges in two ways – either in the same orientation or in opposite orientations (Figure 1a and 1b). By iterating, one can easily form infinitely long combinations that have a sinuous appearance (Figure 1c). The question naturally arises: can we make these chains close up into loops, and if so, in what kinds of ways? If one simply starts building chains using random walks (and avoiding overlaps), it quickly becomes apparent that closing loops is not trivial. One can easily reach a “near miss” situation such as the one shown in Figure 1d. A better strategy is to start from a known loop and define operations that are guaranteed to transform one loop to another. This by no means guarantees the discovery of all possible loops, but it allows rapid exploration and creation of many interesting examples.

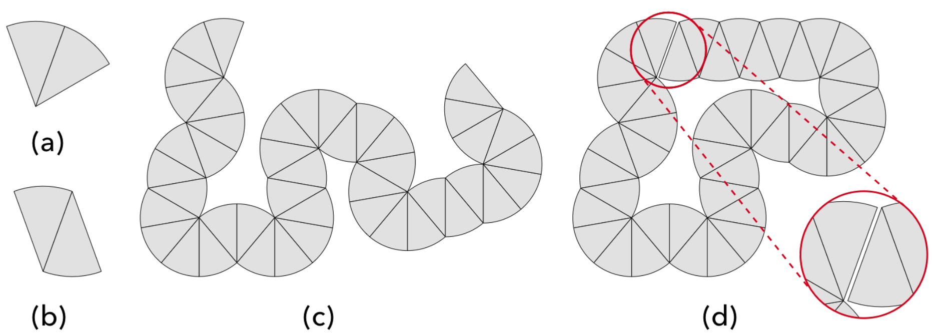

Figure 1: *Sectors in same orientation (a), sectors in opposite orientation (b), a random chain (c), and a near-miss loop with small gap near upper left (d).*

For ease of presentation, we will place one sector with its center line vertical (S_0^n), and we will label the other sectors as $S_1^n, S_2^n, \ldots, S_{n-1}^n$ (Figure 2a). To accommodate opposite pairings, we will take the horizontal mirror image, label the mirror image of S_0^n as T_0^n, and again label the remaining segments as $T_1^n, T_2^n, \ldots, T_{n-1}^n$ (Figure 2b). The reverse labeling is chosen so that oppositely connecting pairs will share the same subscript. If n is understood in a given context, we can omit the superscripts and simply use S_i and T_i, as we will do for the rest of this paper[1]. Also, we will sometimes refer to S_i as the ***clockwise*** sectors and T_i as the ***counterclockwise*** sectors.

[1]Figures in this paper use $n = 9$, but all results here are valid for all $n \geq 3$.

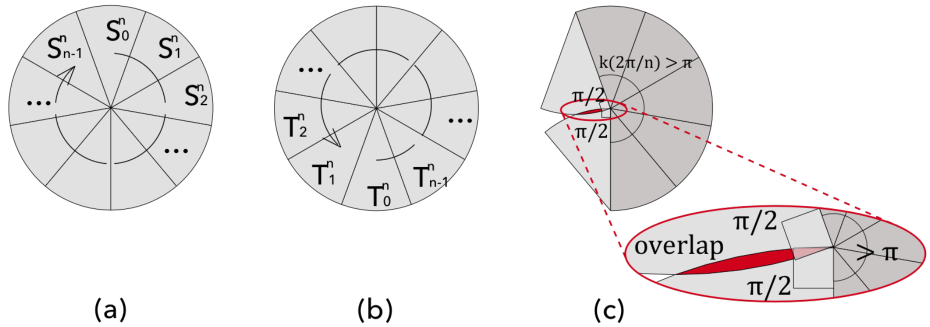

Figure 2: *Clockwise sectors S_i (a), counterclockwise sectors T_i (b), overlap when angle occupied by the sectors in a block exceeds π (c).*

We will use ***chain*** to mean any number of sectors S_i and T_i combined along their radial edges, and a ***simple chain*** is one in which sectors never overlap. By definition, all chains alternate between some number of sectors S_i and T_i. We will call each group of contiguous sectors in the same orientation a ***block***. Since each block is bounded by oppositely-oriented sectors whose arcs emerge perpendicular to the block ends, the total angle occupied by the sectors in a block cannot exceed π or else the bounding blocks will overlap (Figure 2c). Since each sector occupies an angle of π/n, this means the maximum number of contiguous S_i or T_i sectors in a block is $\leq \lfloor n/2 \rfloor$. If we restrict ourselves to blocks of this maximum size, we can now compose chains from these blocks without fear of overlap at the junctures. We can label the blocks with the number of sectors they contain as jS and jT with $j \leq \lfloor n/2 \rfloor$ (Figure 3 top) and adopt a condensed notation for a chain, e.g.: $[3S, 2T, 1S, 3T, 2S, 1T]$ instead of $[S, S, S, T, T, S, T, T, T, S, S, T]$ (Figure 3 bottom). Since exchanging jS and jT blocks merely constitutes a horizontal mirroring of the entire chain, we can further condense our notation to simply $[3, 2, 1, 3, 2, 1]$ without loss of generality. Repeating a given chain $[a, b, c]$ k times can be notated as $[a, b, c, \ldots]^k$.

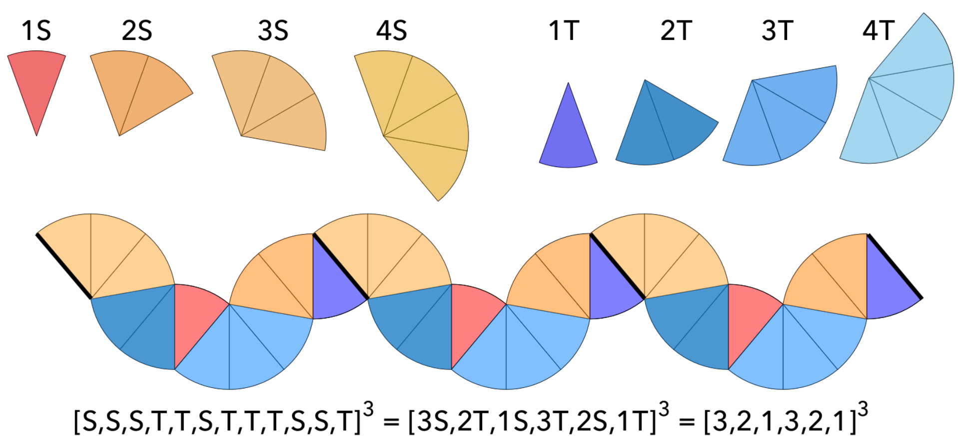

Figure 3: *All valid blocks for n=9 (top); a repeating straight chain with block notation (bottom).*

A chain that arrives back exactly at its starting point, in the same orientation, is a ***loop***, and one that never overlaps itself is a ***simple loop***. We are seeking methods to generate as many simple loops as possible for a given n (ideally, all possible ones).

Straight Chains and Rotationally Symmetric Loops

Before tackling loops, let's consider "straight" chains. We will call a chain ***straight*** if its last radial segment is parallel to its first because its final orientation is in the same direction—despite the fact that its appearance may be quite curvy! It is easy to see that the chains $[a, a]^k$, for $1 \leq a \leq \lfloor n/2 \rfloor$ and $k \geq 1$ are straight. More generally, any chain $[a_1, b_1, a_2, b_2, ..., a_n, b_n]$ will be straight as long as $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. The reason for this is that these sums represent the number of clockwise S sectors and counterclockwise T sectors, respectively. It is equally evident that any two straight chains whose first and last radial segments have the same orientation can be composed into a longer straight chain. If the chain begins and ends with blocks of the same orientation, and their total number exceeds $\lfloor n/2 \rfloor$, then as previously noted there will be overlap. However this is easily overcome by simply inserting one or more pairs of S and T sectors until the overlap is eliminated (in Figure 4 top right, [3,2,1,4,2,4,1,1,4] would overlap when repeated; adding the extra [1,1] solves this problem).

Given any straight chain, two different rotationally symmetric loops can be formed by adding either an S or T sector at the end (introducing a $2\pi/n$ turn to the previously straight chain), and then connecting n such chains. As with composing straight chains, we can insert pairs of S and T sectors to eliminate overlap as needed. Figure 4 shows four successively longer straight chains and the pairs of loops formed in this way.

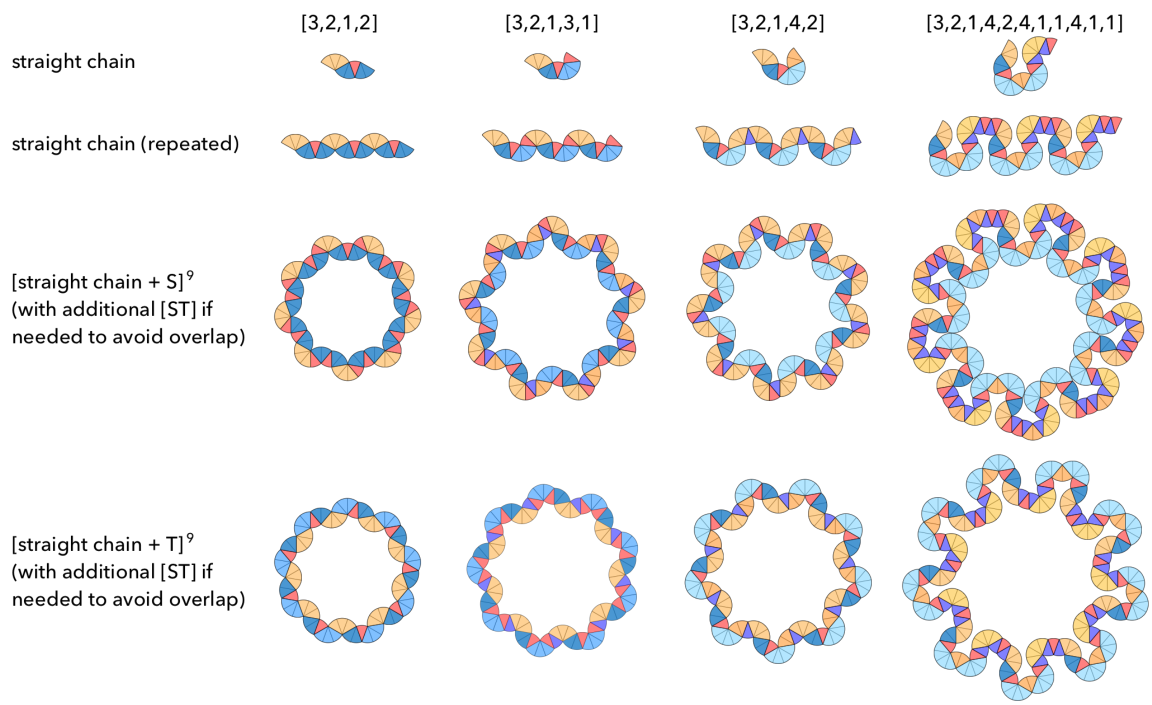

Figure 4: *Top row: four straight chains of increasing curviness. From top to bottom: single chain; repeated 3 times; loop formed by adding one S segment; loop formed by adding one T segment.*

Asymmetric Loops

The method above will only ever yield simple loops with n-fold rotational symmetry. However, in many cases one can create an asymmetric loop by taking a symmetric loop, cutting out pairs of sections along triples of parallel radial edges, and "splicing" the resulting pairs of sections back in the opposite order (Figure 5). As before, any overlaps can be overcome by adding pairs of S and T sectors to the segments of the original loop.

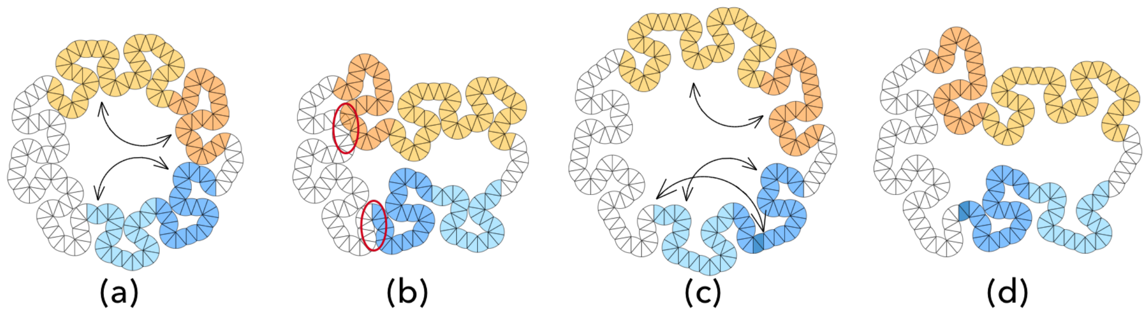

Figure 5: *Starting symmetric simple loop (a); overlaps after initial exchange (b); loop with [1,1] pairs added to each segment (c); final asymmetric simple loop (d).*

Lizardy Loops as a Basis for Artwork

My whole exploration of this topic was motivated by the fact that my existing fractal Islamic patterns based on combined center grids [1] could be divided into sectors and combined in the opposite orientation. Recombining existing patterns led to the example below (Figure 6). Decorating the sectors in different ways could yield endless possibilities for artwork.

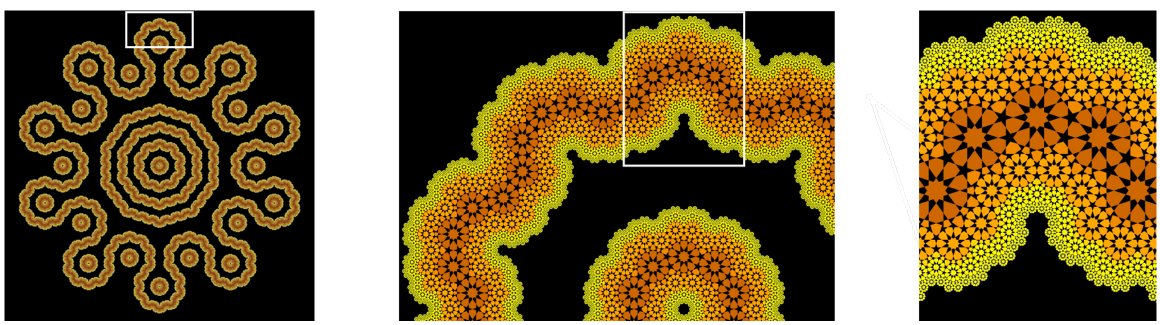

Figure 6: *Fractal Islamic Pattern artwork based on lizardy loops (3 zoom levels).*

Summary and Conclusions

The results here are only an initial look at the problem. Just prior to publication it became obvious that these chains could be analyzed using a vector approach, considering the vector leading from midpoint to midpoint of the radial segments of each sector as its direction. These vectors connect head to tail throughout the chain and would allow the entire problem to be viewed as "turtle geometry in disguise". An analytic approach based on these vectors might predict when chains will self-intersect, when chains will close into loops, and ideally define an algorithm to generate all possible loops. Other possible extensions include mixing sectors for different values of n and replacing circular sectors with polygonal sectors.

References

[1] P. Webster. "Fractal Islamic Geometric Patterns Based on Arrangements of $\{n/2\}$ Stars." *Bridges Conference Proceedings*, Enschede, the Netherlands, July 27–31, 2013, pp. 87–94. http://archive.bridgesmathart.org/2013/bridges2013-87.html