Origami Explorations of Convex Uniform Tilings Through the Lens of Ron Resch's *Linear Flower*

Uyen Nguyen[1] and Ben Fritzson[2]

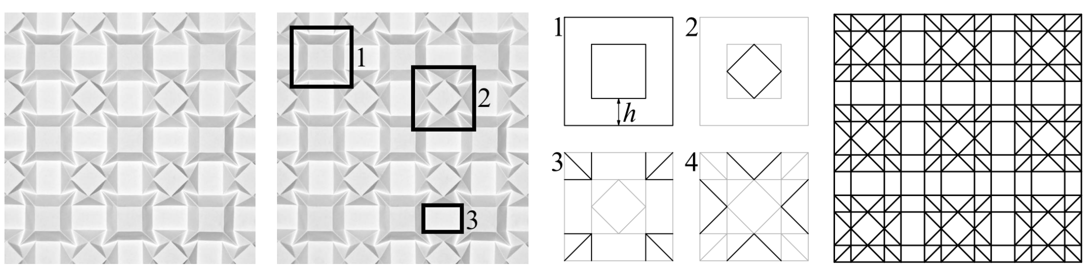
[1] New York, New York, USA; win@winwin.fashion
[2] Philadelphia, Pennsylvania, USA; ben.fritzson@gmail.com

Abstract

We present a method of creating origami tessellations in the style of Ron Resch's *Linear Flower*. We designed a process that allows us to construct a crease pattern modeled after any convex uniform tiling by calculating where folds on that crease pattern should be placed to get a desired result when folded. This has allowed us to move beyond Resch's original square grid and apply it to any Archimedean and k-uniform tiling or their duals.

Introduction

Linear Flower is an origami tessellation by the late Ron Resch [2]. We created a reconstruction of the work (Figure 1a) by estimating proportions from the artist's photographed pieces. There are three basic structures that comprise *Linear Flower* (Figure 1b): 1- simple units with a flat top, rectangular sides, and a leg sloping away from each corner; 2- complex units with a flat top, triangular sides, and indented pockets at the corners which overlap the simple unit's legs; 3- rectangular regions not used by either type of unit, known in origami terms as rivers [1]. We define a reference plane as the surface against which the rivers lie flat.



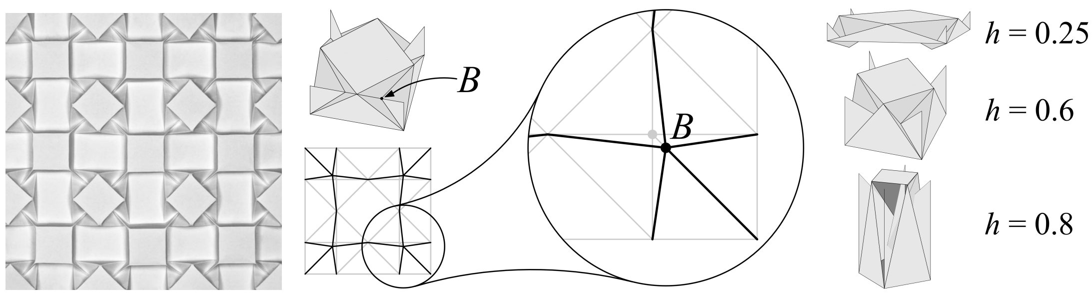
(a) **(b)** **(c)** **(d)**

Figure 1: *The basic structures of Linear Flower*

To generate a preliminary crease pattern (Figure 1c): 1- Start with a polygon and a duplicate of that polygon that has been reduced by an amount h on each side, where h is the height of the folded complex unit. In the case of *Linear Flower*, the polygon is a square and $h = 0.5$ assuming a unit length apothem. 2- Join the midpoints of the reduced polygon to form the top surface of the complex unit. 3- Form the shared legs of the simple and complex units with three lines at each vertex of the reduced polygon: the first connects to the corresponding vertex of the starting polygon, and the other two perpendicularly intersect the sides of the starting polygon. 4- Connect the midpoint polygon to the shared legs to form the walls of the complex unit. This completes the complex unit. The full crease pattern is formed by tiling complex units and connecting orthogonally adjacent units' corresponding vertices (Figure 1d). The amount of space between complex units determines the size of the simple units and the width of the rivers. In the case of *Linear Flower*, the spacing is equal to the edge length of the midpoint polygon.

Modifying *Linear Flower*

In the original *Linear Flower*, Resch's design results in slightly sloped walls and gently splayed pockets. However, when the same process is applied to other shapes, the splaying is often exacerbated, resulting in a model that cannot lie flat against the reference plane when the creases are fully formed. To avoid this, we wanted to fold an idealized version of *Linear Flower* where the sides of the units are perpendicular to the reference plane and the legs are folded completely so the indented pockets are tetrahedral in shape (Figure 2a). Simply squeezing the *Linear Flower* to force this condition causes the legs of the units to buckle, since the pockets are too shallow for the legs to remain straight when completely folded. To achieve this result while avoiding unwanted deformation, we modify the crease pattern by holding all points of the preliminary crease pattern fixed except for point B, the deepest vertex of the tetrahedral pocket (Figure 2b). Shifting point B towards the center of the complex unit produces shallower pockets while shifting it away from the center produces deeper pockets. The calculation for the shift in B is discussed later in this paper.

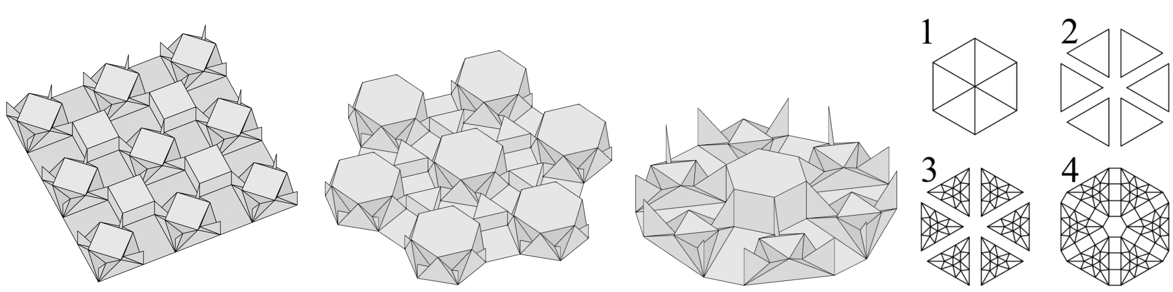


(a) **(b)** **(c)**

Figure 2: *Modifying Linear Flower*

Linear Flower can be further modified by changing the height of the complex unit. Assuming a unit length apothem, h is theoretically bounded by 0, where the paper would remain flat, and 1, where the midpoint polygon shrinks to a point. However, there are physical limits within that range. When $h > 0.6702$, the pockets extend deeply enough into the unit that they collide with one another; when $h < 0.4143$, the pockets dip below the reference plane (Figure 2c). These specific limits only apply to a square complex unit and are found by solving for h when B is touching the reference plane or located at the center of the unit, respectively.

New Constructions



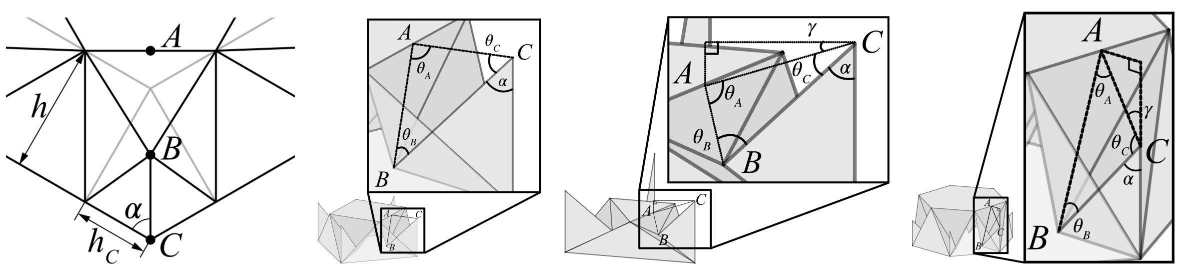
(a) **(b)** **(c)** **(d)**

Figure 3: *Tessellations from regular tilings*

Linear Flower uses squares to create complex units, but we can create new works of art in the same style by applying the same construction methods to different shapes, such as hexagons or triangles. To do so, 1- start with a base tiling; 2- explode the tiles to form rivers; 3- construct complex units in the base tiles, incorporating the shifting of point B; 4- connect the spaces between complex units, which forms the simple units (Figure 3d). The base tiling for *Linear Flower* is the square grid. Since the square grid is a self-dual tiling, both complex units and simple units are square (Figure 3a). Tiling hexagonal complex units produces triangular simple units and vice versa, since the shapes that form the simple units compose the dual tiling of the shapes that form the complex units (Figure 3b and 3c). Furthermore, the heights of the simple and complex units are the same only for square units. Complex units with acute interior angles will produce simple units at a taller height, and those with obtuse interior angles will produce simple units at a lower height.

B Shift Calculation

In our mathematical model, we assume origami does not stretch the paper and all deformation is localized to the folds which act as hinges. Therefore the distance between points on paper will be the same in the folded model as in the flat crease pattern. We determine the shift in B by constraining the parameters of the folded model to agree with that of the flat crease pattern. We start with a preliminary crease pattern for one complex unit generated as shown in Figure 1c, substituting a different value for h and a different polygon as desired. We examine one vertex (Figure 4a). Point C is the vertex of the starting polygon, point A is the location where line $\overline{BC}$ would intersect the midpoint polygon if extended, and α is half the interior angle of the vertex. $\overline{AC}_{\text{flat}}$ remains constant regardless of how far B shifts. The gray lines show the location of point B prior to shifting, which is the vertex of the reduced polygon. When the complex unit is folded, point C is located directly above this vertex. Thus, we can calculate or measure $\overline{AC}_{\parallel}$, the parallel component of $\overline{AC}_{\text{folded}}$ relative to the reference plane.



(a) **(b)** **(c)** **(d)**

Figure 4: *Calculating the B shift*

Now we examine a folded complex unit where point C may be level with, taller than, or lower than point A (Figure 4b-4d). Point C is located at $h_C = \frac{h}{\tan(\alpha)}$ above the reference plane. Subtracting h gives us $\overline{AC}_{\perp}$, the perpendicular component of $\overline{AC}_{\text{folded}}$ relative to the reference plane, thus $\overline{AC}_{\text{folded}} = \sqrt{\overline{AC}_{\parallel}^2 + \overline{AC}_{\perp}^2}$. The angle between points A and C is γ and can be calculated from $\overline{AC}_{\perp}$ and $\overline{AC}_{\parallel}$. Then $\theta_C = 90° - \alpha - \gamma$ if C is taller than A and $\theta_C = 180° - \alpha - \gamma$ if A is taller than C. Using the Law of Sines and software with an equation solver such as MATLAB, solving the following four equations simultaneously gives the modified length of $\overline{BC}$ and point B can be shifted accordingly:

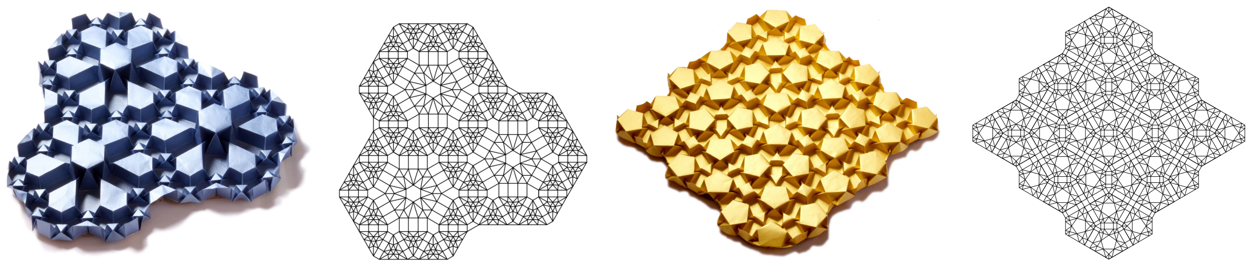
$$\overline{AC}_{\text{folded}} \cdot \sin(\theta_C) = \overline{AB} \cdot \sin(\theta_B) \qquad \overline{BC} \cdot \sin(\theta_C) = \overline{AB} \cdot \sin(\theta_A)$$

$$\theta_A + \theta_B + \theta_C = 180° \qquad \overline{AB} + \overline{BC} = \overline{AC}_{\text{flat}}$$

This result must be calculated for each unique vertex of the complex unit.

Semiregular Tilings

We can design works that use more than one type of complex unit, so long as the base tilings are convex and edge-to-edge. Archimedean and k-uniform tilings serve as good base tilings for such designs. An example is the snub trihexagonal tiling (Figure 5a and 5b) which uses both triangular and hexagonal complex units. The heights of the complex units used are linked. Choosing the height for one determines the height of the simple unit, which in turn determines the height for the other complex unit since both complex units contribute to the formation of the simple unit. The spacing between units is such that the rivers remain rectangular. The resulting simple unit shapes are that of the floret pentagonal tiling, dual to the snub trihexagonal tiling.



(a) **(b)** **(c)** **(d)**

Figure 5: *Origami tessellations folded from Stardream paper (0.16 mm thick) and crease patterns from semiregular tilings*

Irregular polygons can also be tiled in this style if they meet the criterion that every vertex joins only one type of angle (e.g. four 90° angles, three 120° angles, or eight 45° angles coming together). The reason for this is that the height of the simple unit must be uniform for each of the complex units that make up the simple unit. Catalan tilings (Archimedean duals) and k-uniform dual tilings satisfy this condition. Figures 5c and 5d show one such example with a tessellation created from a Cairo pentagonal tiling base.

Future Work

It is possible to create tessellations from irregular tilings even if they do not meet the vertex criterion described above if we modify the method for generating the preliminary crease pattern. Instead of drawing the legs to perpendicularly intersect the starting polygon (Figure 1c), we can draw them a fixed distance away from the unit vertex. This distance is the height of the simple unit, and forcing all the legs to produce the same height ensures that simple units will be able to form. However, this doesn't always work when dissimilar angles are joined at a vertex (e.g. very obtuse and very acute angles joining). Since obtuse angles lend themselves to producing a shorter simple unit and acute angles lend themselves to producing a taller simpler unit, forcing the simple unit height may prevent the tetrahedral pockets from forming properly. We plan to develop methods that fine-tune the parameters to allow for the formation of tessellations from irregular tilings such as rhombic Penrose tilings and Voronoi tessellations.

Acknowledgements

We thank Katrina S. Forest, Robby Kraft, Marcus Michelen, Max Shevertalov, Jennifer Tashman, and Ninh Tran for useful discussions.

References

[1] R. Lang. *Origami Design Secrets: Mathematical Methods for an Ancient Art*. 2nd ed. CRC Press, 2012, pp. 749.

[2] R. Resch. "Periodic Paper Folding or Tessellated Origami." http://www.ronresch.org/ronresch/gallery/extreme-paper/.