Girih Tiles in 3D

Ulrich Reitebuch[1], Henriette-Sophie Lipschütz[2], and Konrad Polthier[3]

Department of Mathematics and Computer Science, Freie Universität Berlin, Germany
[1]ulrich.reitebuch@fu-berlin.de, [2]henriette.lipschuetz@fu-berlin.de, [3]konrad.polthier@fu-berlin.de

Abstract

In Islamic architecture and art, girih patterns are used for decorations of buildings such as mosques or palaces. Based on the construction of 2D girih patterns, we provide a corresponding procedure to construct 3D girih patterns covering $\mathbb{R}^3$ completely.

Classical 2D Girih Tiles

Girih patterns are known from decorations of various buildings in the style of Islamic architecture like the Dome of the Rock in Israel or the Topkapı Palace in Turkey. Classical girih patterns arise from girih tiles that consist of five equilateral polygons–a regular decagon, an irregular convex hexagon, a non-convex hexagon, a rhombus, and a regular pentagon, see Figure 1a. The girih patterns are created by piecewise straight lines that cross in the midpoint of an edge of a girih tile and form an angle of $\frac{3\pi}{10}$ with the edge.

A more general construction results from a suitable polygonal tiling. Again, straight lines are added at the edge midpoints with a chosen angle to the edge such that they continue over the edge into the neighboring tile. The intersection points of the lines belonging to one tile are determined and for each line ordered by distance to the edge midpoint. The lines are grouped into intersecting pairs. Then both lines of every pair are cut off at their intersection point. Hence, there arise sequences of straight line segments. For regular tiles, selecting the first or second intersection points gives such a pairing of lines, see Figure 1c. These lines might be used directly as decoration, in that case often with alternating above resp. below crossings to create woven lines, or the areas bounded by the line segments are colored differently. Usually, the start girih tiles are used as a support structure and in the final decoration they are invisible. The construction is illustrated in Figure 1.

This procedure creates beautiful patterns consisting of lots of (regular) stars surrounded by a chain of faces containing the corners of the original tiling.

A nice example is the 8-4 tiling that is discussed by J. Bourgoin and E. H. Hankin [1, 3].

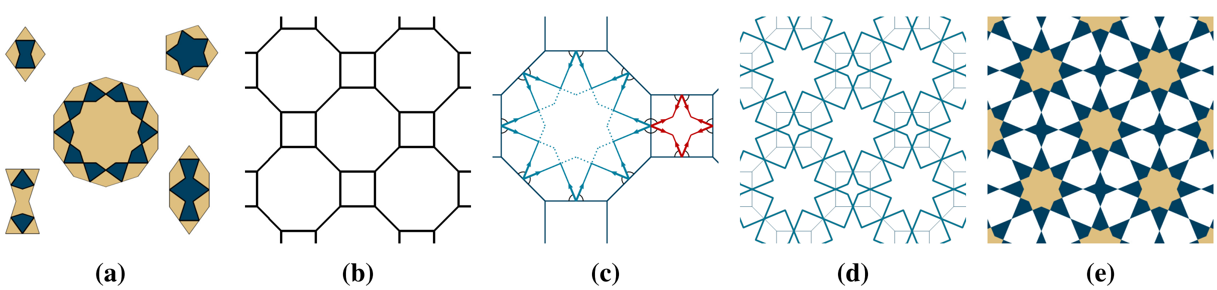
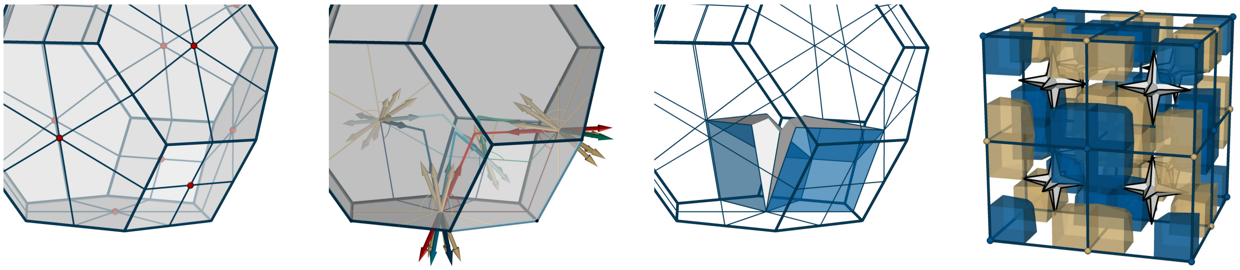


Figure 1: *Construction of classical girih decorations: (a) classical girih tiles; (b) tiling built from squares and regular octagons; (c) girih lines drawn with a chosen angle from edge midpoints until they intersect; (d) line pattern with first intersection points in octagons; (e) pattern with second intersection points in octagons and with differently colored areas.*



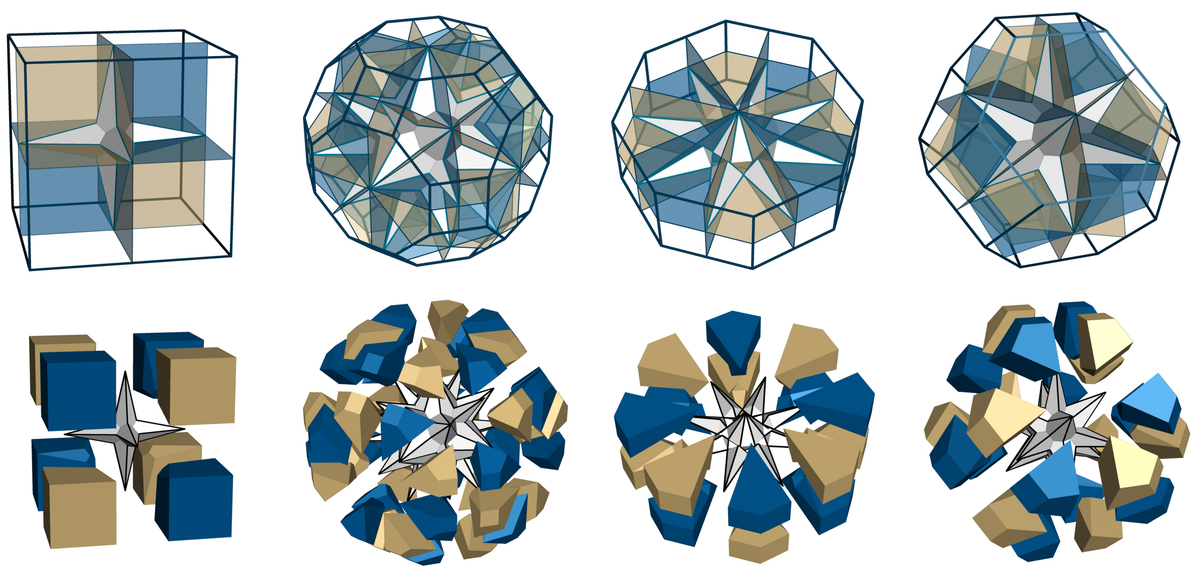
(a) **(b)** **(c)** **(d)**

Figure 2: *Construction of a solid in a 3D tiling: (a) perpendicular bisectors in all faces; (b) arrows respresenting constructed 3D girih lines; (c) constructed faces where the shaded faces arise from neighboring 3D girih lines while blue quadrilaterals connect lines of neighboring faces; (d) joining corner solids according to 2-coloring.*

The knowledge of the construction of girih patterns and tiles is being handed down in the Topkapı Scroll (presumably from the end of the 15th century or beginning of the 16th century). The patterns contained in the scroll are presented and commented by G. Necipoğlu [4].

Extending to 3D Tilings

Analogously to the 2D case described above, we choose the same approach to construct a 3D tiling in girih style. We take a 3D tiling of space consisting of solids with regular faces only, such as the dense covering with cubes or–corresponding to the 8-4 tiling in 2D–a tiling consisting of equilateral octogon prisms and cubes.



(a) **(b)** **(c)** **(d)**

Figure 3: *First line: illustration of the construction of faces limiting the star-shaped solid and the corner solids; second line: star-shaped solids and corner solids created from the four basic tiles; (a) cube; (b) truncated cuboctahedron; (c) octogon prism; (d) truncated octahedron.*

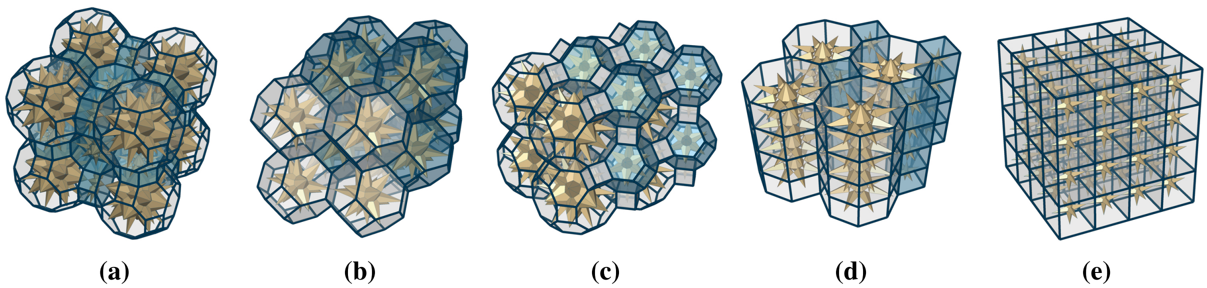


Figure 4: *Construction of 3D tilings in girih style: tilings built from (a) octagon prisms and truncated cuboctahedra; (b) truncated octahera; (c) cubes, truncated octahedra, and truncated cuboctahedra; (d) octogon prisms and cubes; (e) cubes.*

Every 3D tile will be split into one star-shaped region and several small solids at the corners of the tile. All small solids belonging to different tiles sharing an original corner are joined to one solid containing that corner. Hence the original tiling is no longer visible, as in the 2D case, see Figure 1. Instead of edge midpoints in 2D, now the face midpoints are chosen. They are chosen as the intersection of the perpendicular bisectors of the boundary edges, see Figure 2a. Since the faces are regular the bisectors intersect in exactly one point. To construct the star-shaped solids we choose a *3D girih line* in every plane spanned by an edge bisector and the face normal. These lines pass through the face midpoint and form a given angle with the face, see Figure 2b.

The faces of the tiling elements arise from the 3D girih lines as follows: for every face midpoint, each pair of neighboring 3D girih lines span a plane. In every tile, all those planes together bound a star-shaped solid. The perpendicular bisectors and the corresponding 3D girih lines bound a quadrilateral. These quadrilaterals divide the remaining part of the solid into smaller solids each containing one of the original corners.

The first and easiest suitable class of solids for 3D tiling are built with regular faces only, with even number of vertices incident to each face. The requirement of an even number of vertices is necessary since otherwise the 3D girih lines do not align with the lines in the neighboring solid. The edge graph of the

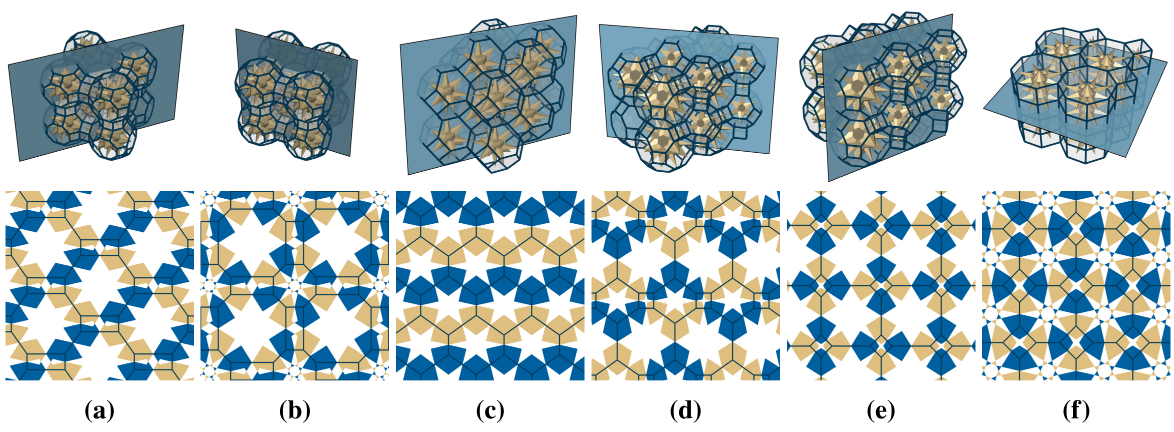


Figure 5: *Tilings resulting from intersection with planes parallel to the axis resp. parallel to an axis and a face diagonal: obtained by (a) diagonal intersecting 4a; (b) parallel intersecting 4a; (c) diagonal intersecting 4b; (d) diagonal intersecting 4c; (e) parallel intersecting 4c; (f) parallel intersecting 4d.*

original tiling is 2-colorable since it contains only even cycles. This allows a 2-coloring of the corners, so the joined new solids around the corners of the original tiling can be 2-colored as well. Thus we get three sets of new solids, namely the center star in each solid and the two sets of corner solids. Examples of composed corner solids can be seen in Figure 2d and in Figure 6.

Figure 3 illustrates the small solids resulting from the procedure described above based on the use of the following tiles: cube, truncated cuboctahedron, octogonal prism, and truncated octahedron.

In the case of the octogonal prism we do not use the first intersection point of the 3D girih lines but the second intersection point due to aesthetic reasons.

There are five different tilings of space consisting of the listed tiles: the first consists of octogon prisms and truncated cuboctahedra, the second contains truncated octahedra, the third uses cubes, truncated cuboctahedra, and truncated octahedra, the fourth is built of octogon prisms and cubes, and the fifth is made of cubes only, see Figure 4 and [2].

Moreover, intersecting the newly constructed 3D tilings in girih style with planes parallel to the axis resp. parallel to an axis and a face diagonal result in tilings very similar to those observed in the 2D case, see Figure 1 and Figure 5.

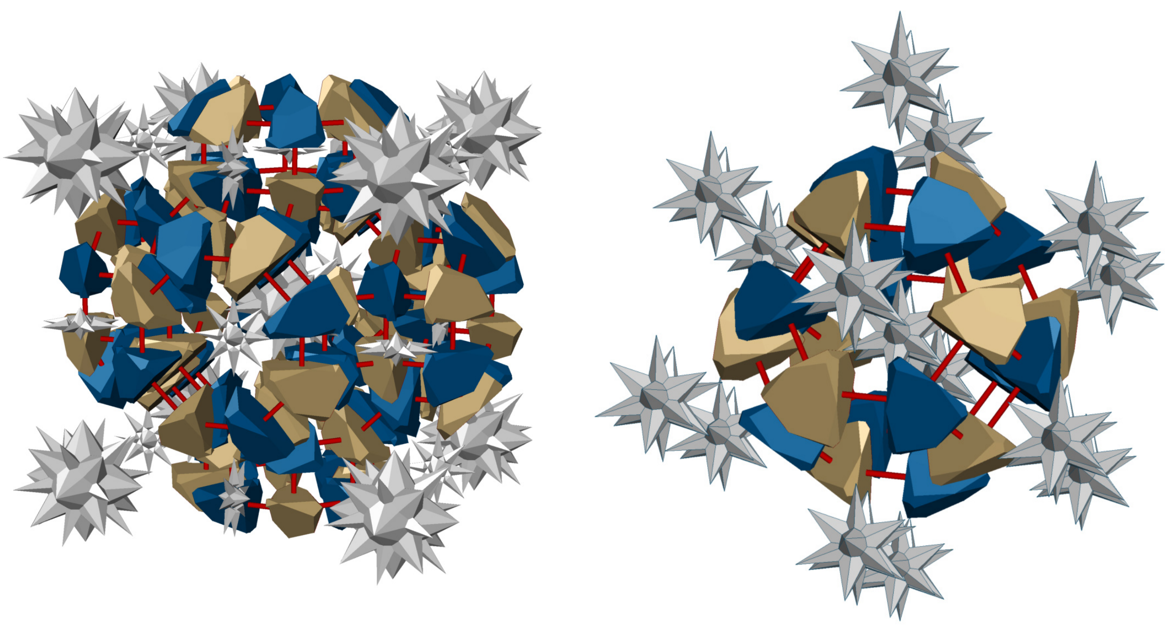


Figure 6: *Exploded assembly drawings of the tilings 4a and 4b.*

References

[1] J. Bourgoin. *Les Éléments de l'Art Arabe*. Paris: Firmin-Didot et cie, 1879.

[2] J. H. Conway, H. Burgiel, and C. Goodman-Strauss. *The Symmetries of Things*. Wellesley, MA, USA: A. K. PETERS, Ltd., 2008.

[3] E. H. Hankin. *The Drawing of Geometric Patterns in Saracenic Art*. Archaeological survey of India: Memoirs. Government of India, Central Publication Branch, 1925.

[4] G. Necipoğlu et al. *The Topkapı Scroll: Geometry and Ornament in Islamic Architecture : Topkapı Palace Museum Library MS*. Sketchbooks & albums v. 1956. Getty Center for the History of Art and the Humanities, 1995.