A Design Method Based on Close-Packing Circles and Spheres of Multiple Sizes for Designers and Architects

Roger Burrows

5621 Bobsled Blvd., Park City, Utah 84098, USA; rogerburrows@me.com

Abstract

A design method is presented that generates many new close-packing circle and sphere arrangements of multiple sizes within tessellating tiles and prisms from which two and three dimensional polygonal shapes can be extracted.

Introduction

The story begins with a window design in the Sarghatmish Mosque Madrasa (1356 CE) in Old Cairo, Egypt, Figure 1, and an interpretation of the logic used to create the design on page 163 of "L'Art Arabe" (1879 CE) by Jules Bourgoin, see [1] and Figure 2(a). Bourgoin's work focused on determining Islamic geometric methods used during the Abbasid Caliphate (750–1258CE in Baghdad and 1261-1517 CE in Cairo).

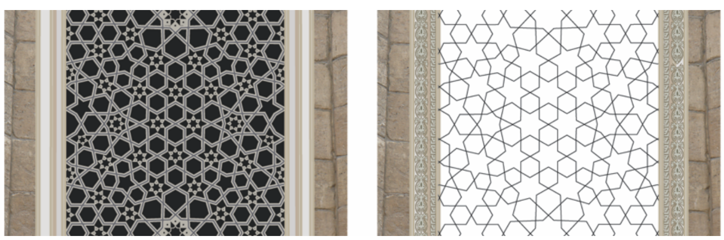

Figure 1: *Line drawings of the window in the Sarghatmish Mosque Madrasa (1356 CE) in Old Cairo.*

The Bourgoin method takes advantage of a near alignment of the lines of symmetry of four regular polygons – an octagon, a hexagon, a pentagon, and a heptagon – within a 45-degree right triangular tile that, when reflected and rotated, creates a tessellating p4m square tile, Figures 2(a) and 2(b). Bourgoin positions the centers of four circles on the centers of the lines of symmetry of the polygons to determine the sizes and positions of four rosettes, Figure 2(c). The circles suggest a second interpretation of the design logic – that of five touching circles, see Figure 2(d). For references to Abbasid geometric design methods see [2].

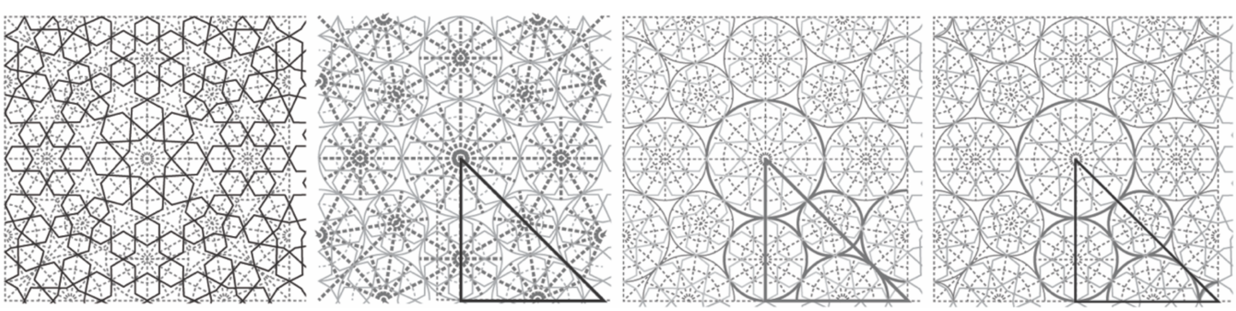

Figure 2: *Figure 2(a) shows the design as it appears on page 163 of "L'Art Arabe;" Figure 2(b) shows the lines of symmetry; Figure 2(c) shows four touching circles; Figure 2(d) shows five touching circles*

Abbasid Close-Packing Circles

Geometrically Abbasid "close-packing" circle arrangements are used as a means to polygonally subdivide two-dimensional space where circles of equal or varying sizes are arranged to touch each other in tangential arrangements, and in triples, without overlapping on a two-dimensional Euclidean plane and positioned symmetrically. The arrangements are typically contained within the bounding mirror lines of "unit" triangular tiles of regular hexagons and squares such as the 45º right triangles of a regular p4m square tiling or the 60º-30º degree right triangles of a regular p6m hexagon tiling, see [2]. The close-packing arrangements are precise – a slight change in circle size or position will negate the close-packing. The branch of mathematics known as "circle-packing" does overlap with Abbasid close-packing but can be differentiated by the fact that it is concerned with the geometry and combinatorics of packings of arbitrarily-sized circles rather than the more specific Abbasid concern – that of generating polygonal forms from close-packing circle arrangements within regular and semi-regular tessellating tiles. For a reference to the branch of mathematics known as "circle-packing," see Kenneth Stephenson's "Introduction to Circle Packing" [6].

Creating Altair Design Two-Dimensional Polygonal Line Patterns

Altair Design Pattern Pad: Bk. 1 line patterns, see [4] and [5], are created by connecting the centers, drawing clipped tangents (to create regular octagons and squares, and "almost" regular pentagons, hexagons, and heptagons), internal polygons and stars at the contact points of the Sarghatmish 5-circles.

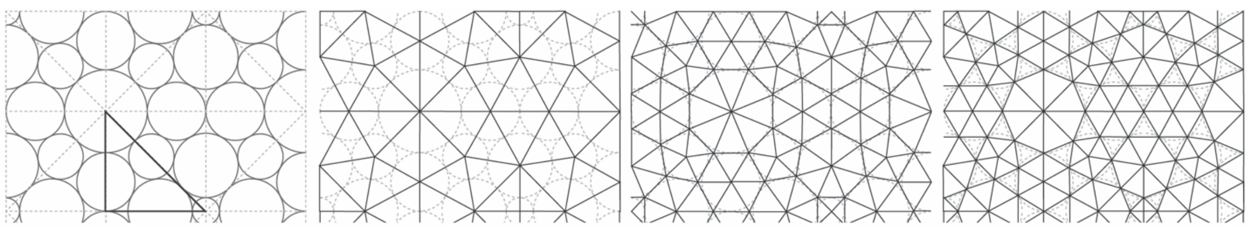

Figure 3: *Figure 3(a) shows the* Sarghatmish *5-circle arrangement; Figure 3(b) connected circle centers; Figure 3(c) clipped tangents at circle contact points with radii to the polygons; Figure 3(d) internal polygons at circle contact points with lines that connect circle centers.*

The Altair line patterns create a "perceptual" effect due to the number of line intersection points where, when contemplating the designs, polygonal shapes seem to appear and disappear before the "mind's eye." The line patterns are also used, Escher like, to abstract shapes that are then reflected, rotated, and translated to create patterns. Users of the line patterns "see" centered designs, flowers, animals, abstract shapes and, in some cases, complete scenes. The perceptual effect is also visible in the original Sarghatmish window.

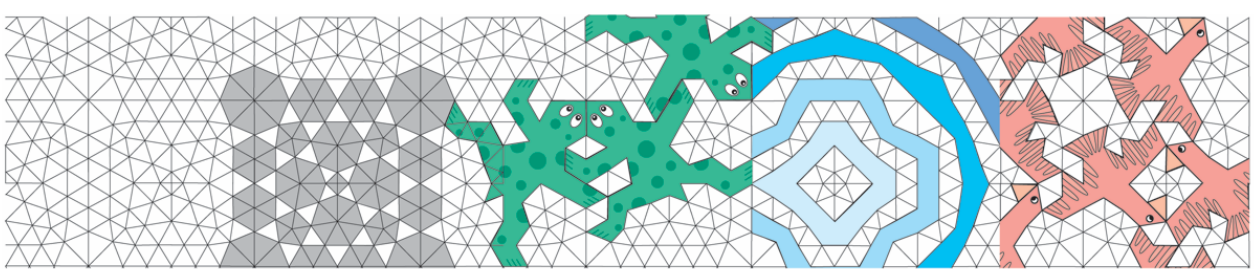

Figure 4: *Polygonal shapes perceived in Altair line pattern 3(d).*

A question arose as to how the 5-circle Sarghatmish close-packing arrangement might have been generated if it was in fact the method used. Working with Dr. Ensor Holiday in the early 1970s the author developed a step-by-step logic that started with a basic arrangement of five circles within a 45-degree right triangle and then allowed the circles to incrementally increase or decrease in size and move position until new close-packing arrangements were generated according to an initial set of parameters, see Figure 5. The third in the generated sequence of Figure 5 is the Sarghatmish arrangement. For animations of close-packing circle and sphere sequences and generated polygonal and polyhedral forms, see [3].

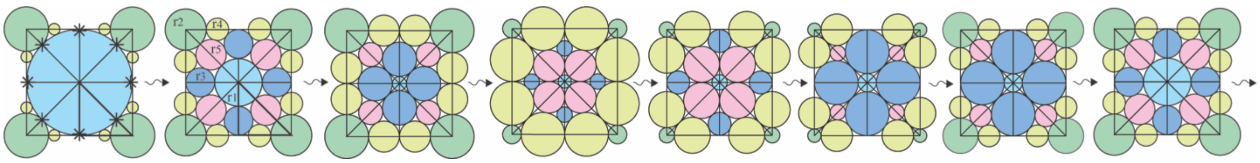

Figure 5: *First eight (of seventeen) 5-Circle close-packing arrangements within 45-degree right triangles. The wavy arrows indicate transition periods where circles do-not close-pack.*

Extending the Two-Dimensional Logic

Extending the Sarghatmish close-packing circle logic began with changing the parameters to that of various containing symmetries and circle counts. One intriguing close-packing sequence was based on a 4-circle rectangular arrangement where circles were allowed to incrementally increase and decrease in size and move along the lines of symmetry, see Figure 6. Other limitations imposed for the sequence were that a central circle, r1, was always to stay in contact with one set of opposite lines of symmetry of the containing rectangle and that circles r3, and then r4 were given a priority of growth. What was of particular interest was that the circles in the second close-packing arrangement were in whole number relationships, corresponding with Soddy's "Bowl of Integers," and that the sixth arrangement generated, exactly, the "Golden Rectangle," ($\sqrt{5}$ +1) / 2 – a precise arrangement of three circle sizes of r1 = 2, r4 = 1, and r3 = ($\sqrt{5}$ – 1).

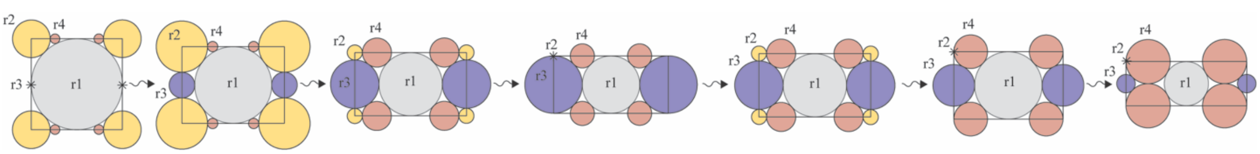

Figure 6: *A 4-circle sequence that starts with a square arrangement where r3 = 0. r3 is given the priority of growth reaching a maximum by the 4th stage. Circle r4 is then given the priority up to the 7th stage.*

Initial applications revolved around the perceptual, and visual logic, properties of the line patterns. However, circular arrangements can be considered as cross-sections of cylindrical arrangements or as the circular bases of hemispherical domes and may well be of use for various types of two and three-dimensional packing. Applications in three-dimensions, see the next section, can be in the areas of architecture where all extracted polyhedral forms will be in proportion, one with another – in this regard the discovery of the "Golden Ratio" close-packing is of interest as a proportional basis for two- and three-dimensional art and design such as recommended in the work of the architect Le Corbusier.

Extending the logic into Three-Dimensions

Extending the logic into three-dimensions was initially based on generating symmetrical sphere arrangements within rectangular hexagonal and equilateral triangular prisms, see [2]. Figure 7(a) shows an arrangement of nine equal sized spheres where r1 = r3 = 2, with internal spheres r4 = 2(√2 - 1) within an equilateral triangular prism. Figure 7(b) shows the change of relative size of r3 and r4 after the step-by-step algorithmic process where r1 = 2; r3 = √5 – 1; r4 = 1; The side proportions of the equilateral prism arrangement are now in the "Golden Ratio," (√5 + 1)/2, see the sixth circle arrangement of Figure 6. In this arrangement six spheres of r4 exactly rotate around r1 and r3. An internal new sphere, r5, is added = √(7/3) - 1 to complete the close-packing. Figure 7(c) shows a 2-layer repeat of 7(b) around a central axis. Figure7(d) shows a structure of mini-triangular surfaces created by connecting the sphere centers of 7(c).

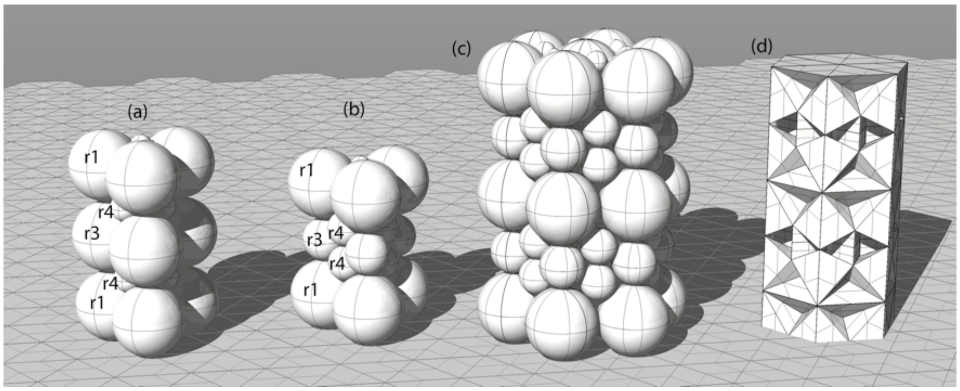

Figure 7: *Shows the result of a step-by-step change in sphere size and position from 7(a) to 7(b); 7(c) is an assemblage of twelve repeats of 7(b); 7(d) is a structure of connected sphere centers of 7(c).*

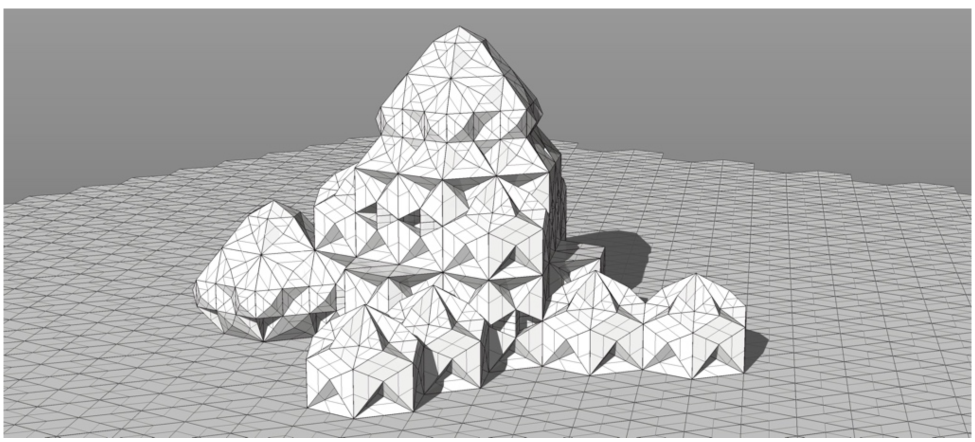

Figure 8: *Structures extracted from repeats of the mini-triangular surfaces of Figure 7(d).*

References

[1] J. Bourgoin. *Les Arts Arabs*. https://archive.org/details/LesElementsDeLArtArabeBourgoin. 1879.
[2] R. Burrows. *3D Thinking in Design and Architecture from Antiquity to the Present.* Thames and Hudson Ltd., 2018.
[3] R. Burrows. Animations. http://rogerburrowsimages.com/think-3d/supplement/ pp. 271-293.
[4] E. Holiday, R. Burrows. *Altair Design Pattern Pad: Bk 1.* Wooden Books, Bloomsbury, 2009.
[5] E. Holiday, R. Burrows, R. Penrose. *Crystal Cave.* Wooden Books, Bloomsbury, 2015.
[6] K. Stephenson. *Introduction to Circle Packing: The Theory of Discrete Analytical Functions.* Cambridge University Press, 2005.