A Threefold Möbius Band with Constant Twist and Minimal Bending as the Limit of Tetrahedral Rings

Johannes Schönke, Michael Grunwald, and Eliot Fried

Okinawa Institute of Science and Technology, Okinawa, Japan
johannes.schoenke@oist.jp, michael.grunwald@oist.jp, eliot.fried@oist.jp

Abstract

We present a previously undiscovered threefold Möbius band which is the unique surface arising from the limit of a sequence of tetrahedral rings, all connected with a minimal, uniformly distributed twist angle. The band is optimal in the sense that its bending is minimized and its midline has the smallest possible total curvature for a closed curve of given fixed nonzero torsion.

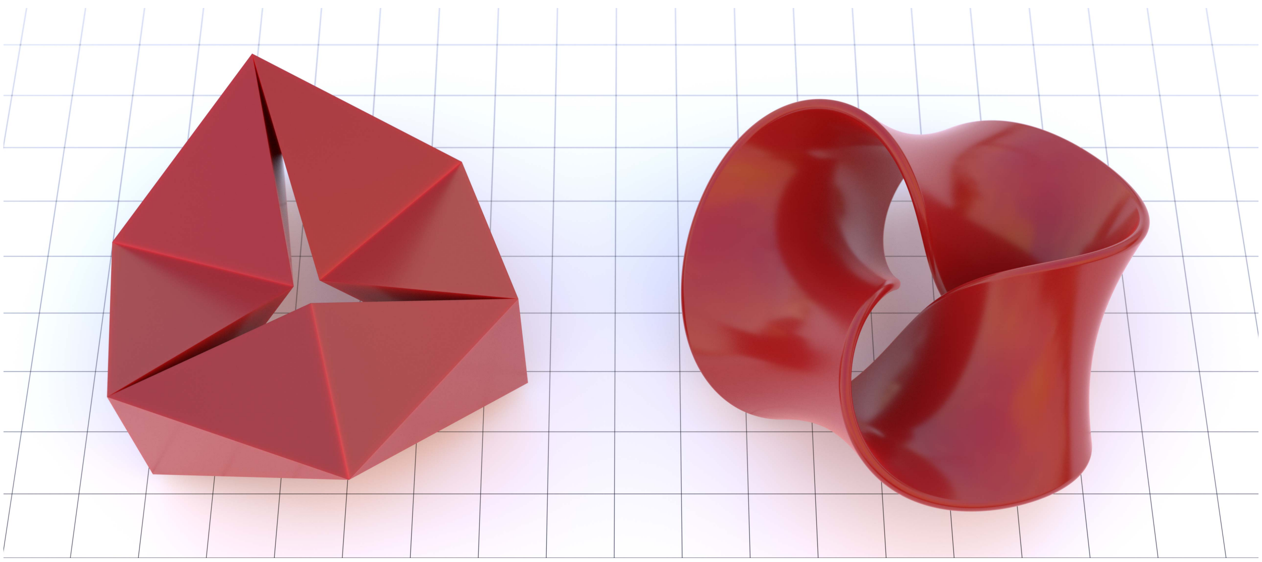

Figure 1: *The classical sixfold kaleidocycle made from six tetrahedra (left) and the threefold Möbius band (right) are related through a sequence of minimally twisted tetrahedral rings. The esthetically pleasing appearance of the band is due to its uniformly distributed twist and its minimal bending.*

Rings of Linked Polyhedra

Consider a chain of N identical polyhedra. Several intriguing question immediately arise: Under what conditions is it possible to close the chain to form a ring? If so, in how many ways can it be done? If the polyhedral volumes are not allowed to overlap, is it still possible to form a ring? If so, how many ways of doing so remain?

We begin with a simple example. Consider a chain of regular tetrahedra with two opposing (orthogonal) edges being the linking sites. If the volumes of the tetrahedra are allowed to overlap, then two such tetrahedra can be linked trivially with their positions being exactly coincident. But there is no way to link three such tetrahedra (Figure 2). Without overlapping volumes, the chain must have at least six regular tetrahedra to form a closed ring. Also, the smallest odd number of tetrahedra needed to form a closed ring is seven, and without volume overlap the smallest odd number of tetrahedra is eleven.

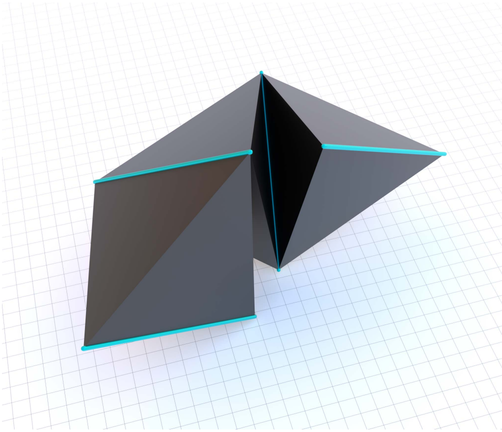

Figure 2: *Three tetrahedra cannot be linked to form a ring even if their volumes overlap.*

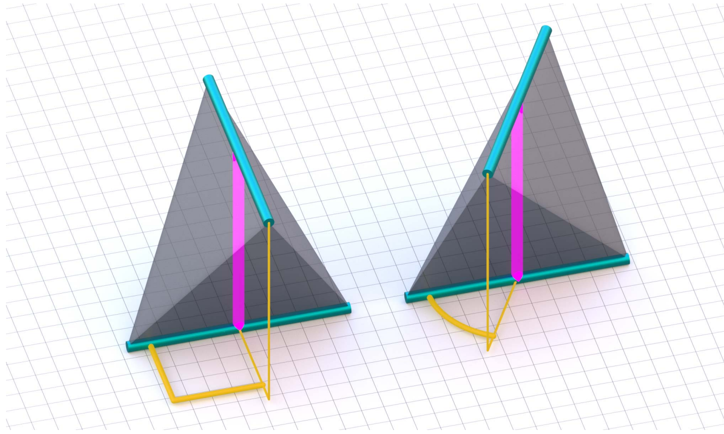

Figure 3: *Tetrahedra with hinges (cyan), midaxes (magenta), and twist angles (yellow) indicated. Left: Regular tetrahedron with orthogonal hinges. Right: Tetrahedron with twist angle $\pi/3$.*

Rings of Linked Twisted Tetrahedra

We next consider "twisted tetrahedra", also called disphenoids, which are tetrahedra whose opposing edges have the same length. As the linking sites we again use two opposing edges which are not generally orthogonal, as illustrated in Figure 3. We refer to the angle between those edges as the twist angle $0 < \alpha \leq \pi/2$. (The singular trivial case $\alpha = 0$, which corresponds to a chain of linked rectangular plates, is excluded from consideration.) Granted that the volumes of the tetrahedra must not overlap (which we require from this point onward), for what values of α is it possible to form a closed ring of N such identical twisted tetrahedra? For $N = 6$ the answer has been known for a long time: It works with six untwisted ($\alpha = \pi/2$) tetrahedra. The resulting object is known as the sixfold kaleidocycle $K6$ (Figure 1, left). The fact that $K6$ — and indeed all the rings we describe below — possess an internal degree of freedom in the sense of a mechanism is so fascinating that we pursue it in a separate workshop contribution to this year's Bridges conference.

For $N > 6$ it turns out that there are many possible twist angles α for which closed rings can be formed. Therefore, we pose the following problem: *For N twisted tetrahedra find the smallest possible α for which a closed ring can be formed.* It turns out that for each $N > 6$ there exists a critical twist angle $\alpha_c \neq \pi/2$ below which one class of closed rings cannot be formed. The determination of the critical twist angle hinges on the real solvability of a system of quadratic equations. We provide a few exemplary values in Table 1 and note that α_c as well as the total twist $N\alpha_c$ are strictly decreasing functions of N. Moreover, for $N \to \infty$ it appears that α_c tends to zero while $N\alpha_c$ approaches a nonzero asymptotic value. As shown in Figure 4, the sequence $KN, N = 6, 7, 8, \ldots,$ of kaleidocycles tends to a surface with the shape of a threefold Möbius band as $N \to \infty$. Note that for the limit object to have a finite size the length of the tetrahedral midaxes (magenta in Figure 3) must also tend to zero. For a discrete equivalent of a band we define the two "sides" of a tetrahedron to be the two pairs of triangular faces which make the larger solid angle along the edges which are not links. (The red central stripes in Figure 4 connect the pairs of triangular faces which constitute one side of a tetrahedron.)

Table 1: *Critical twist angle α_c and total twist angle $N\alpha_c$ for a ring of N twisted tetrahedra.*

N	6	7	8	9	10	11	12	15	21	33
α_c/π	0.5	0.4046	0.3443	0.3010	0.2680	0.2418	0.2204	0.1746	0.1237	0.0783
$N\alpha_c/\pi$	3	2.8319	2.7541	2.7091	2.6800	2.6598	2.6452	2.6189	2.5974	2.5847

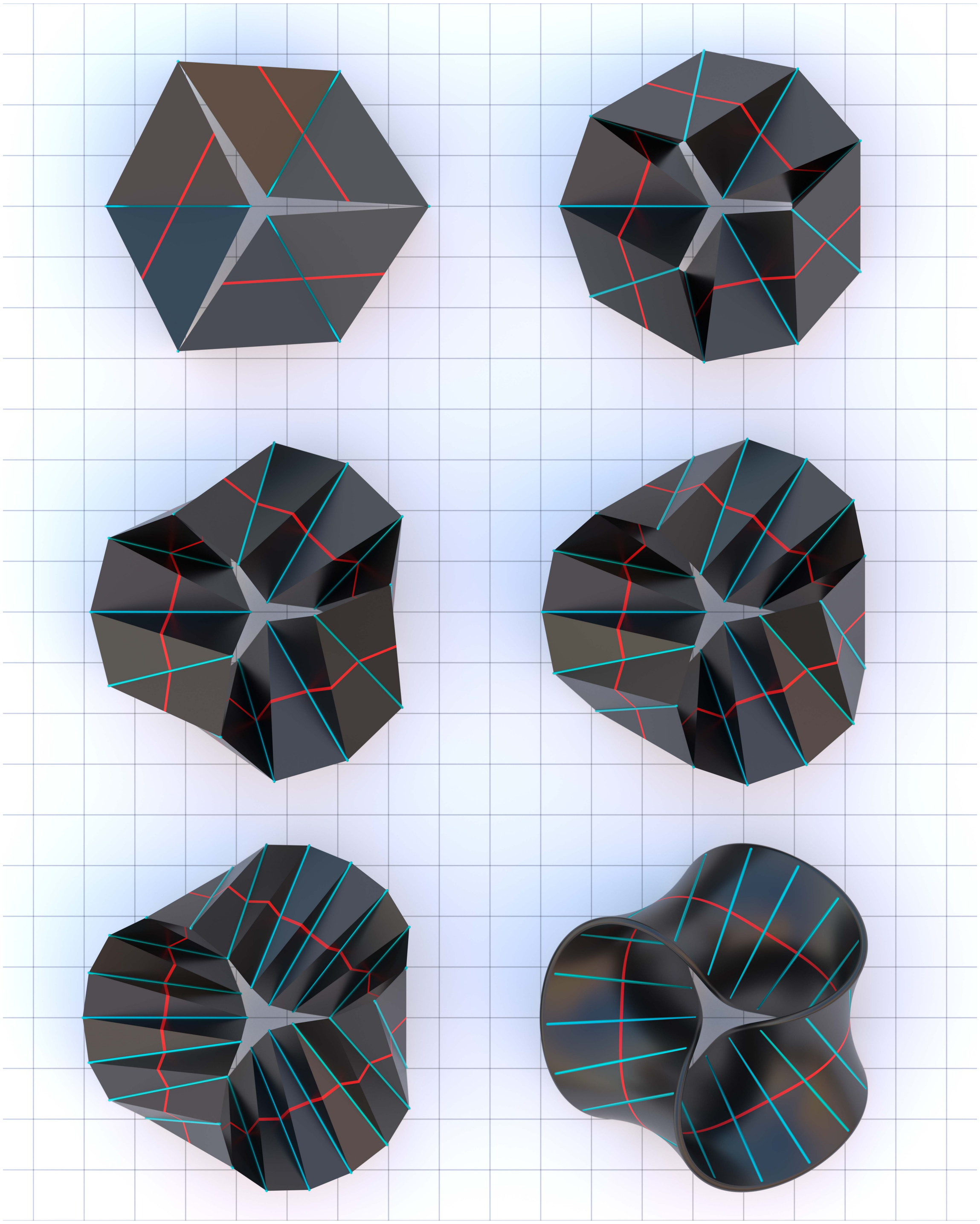

Figure 4: *The limit process of linking an increasing number N of minimally twisted tetrahedra together to form a kaleidocycle KN. Shown are $K6$ (top left), $K9$ (top right), $K12$ (middle left), $K15$ (middle right), $K21$ (bottom left), and the limit surface S for $N \to \infty$ (bottom right), a threefold Möbius band. The linking edges (and the rulings in the limit) are shown in cyan.*

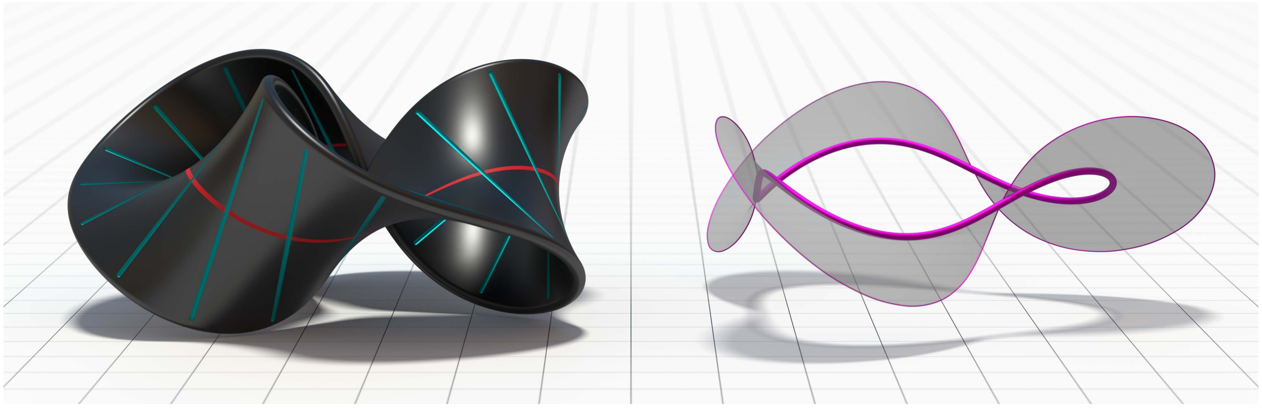

Figure 5: *The limit surface S (left) and its midline curve C with indicated normal curvature (right).*

The Limit Surface and Limit Curve

It is remarkable that the limit surface S shown in Figure 4 and 5 has the topology of a threefold Möbius band, a band formed from a rectangle whose ends are "glued" together after three half turns, instead of the most commonly depicted situation in which the ends are glued after one half turn. Related to this is a threefold rotational symmetry which is clearly visible in Figure 4. By construction, it is clear that S is ruled — the linking edges in the chain become the rulings in the smooth limit surface — but it is not a developable surface, as its twist induces a Gaussian curvature that is everywhere negative. Our numerical results show that *the surface shape resulting from the described limit process is unique* up to the width of the band which is set by the extension of S along the rulings. Beyond a certain width the surface S will begin to self intersect in the center. This is analogous to what occurs when attempting to close a chain of tetrahedra with linking edges that are too long.

The unique closed midline C of S shown in Figure 5 is defined through the limit of the polygonal chain given by the tetrahedral midaxes (Figure 3). Let $\boldsymbol{\gamma}(\sigma)$ be an arc length parametrization of C with arc length $\sigma \in [0, 1]$. The unit tangent $\boldsymbol{t} = \boldsymbol{\gamma}'$ (where a prime denotes the derivative with respect to arc length) to C describes the local rotation axis of the rulings of S. By construction from the twisted tetrahedra it follows that the torsion τ of C is uniform: $\tau = \tau_0$. Our numerical investigations of C suggest that its normal curvature κ is sinusoidal; more specifically, $\kappa(\sigma) = \kappa_0|\sin(3\pi\sigma)|$, with κ_0 being constant. If C has length ℓ, we find that $\kappa_0 \approx 13.022554/\ell$, $\tau_0 \approx 8.094078/\ell$, and the total torsion $\lim_{N\to\infty} N\alpha_c = \tau_0\ell \approx 2.576425\pi$. Up to a scaling C is a unique space curve. Moreover, our calculations indicate that it has novel extremal properties. The problem of determining greatest lower bounds for the total curvature $K(\Gamma) = \int_\Gamma \kappa \, d\sigma$ and the total absolute torsion $T(\Gamma) = \int_\Gamma |\tau| \, d\sigma$ of an arbitrary closed space curve Γ has a long history. See, for example, Fenchel [1]. Although Weiner [2] has shown that a closed curve Γ of constant torsion can have $K(\Gamma)$ arbitrarily close to 2π, the construction underlying his proof is based on a degenerate limit in which $K(\Gamma) \to 2\pi$ and $T(\Gamma) \to 0$ simultaneously, the limit curve being circular. A question relevant to the midline C of S concerns the value of $K(\Gamma)$ for all curves Γ of *nonzero* constant torsion? In this regard, a potentially interesting quantity is the ratio $\Omega(C) = K(C)/T(C) = 2\kappa_0/\pi\tau_0 \approx 1.02426$. Assuming, without proof, that C yields the greatest lower bound of $K(\Gamma)$ for a given fixed nonzero torsion, we are led to conjecture that:

For every closed curve Γ, $\Omega(\Gamma) \geq \Omega(C)$.

References

[1] W. Fenchel. "On the Differential Geometry of Closed Space Curves." *Bull. Amer. Math. Soc. 57*, 1951, pp. 44–54.

[2] J. L. Weiner. "Closed Curves of Constant Torsion." *Arch. Math. (Basel) 25*, 1974, pp. 313–317.