To the World's End/ A Circle Bundle Over a Circle

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Abstract

The sculpture *To the World's End/ A Circle Bundle Over a Circle* is an illustration of the geometry of the Klein bottle and the operation of blowing up in algebraic geometry. An image is featured in Loring Tu's new book *Differential Geometry*.

Figure 1: *To the World's End/ A Circle Bundle Over a Circle*
Lun-Yi Tsai and Zachary Treisman, 2010
Stainless Steel and Latex, 12"×12"×24"

Motivation

Birational transformations, such as blowing up a point on a surface, are central to algebraic geometry. This sculpture emerged out of some musings about latitude and longitude, how these coordinates are quite reasonable for describing a rotating sphere, but they don't exactly work at the poles. Latitude is relevant—the north pole is at $90°N$ and the south pole is at $90°S$. But longitude at the poles specifies not a location but a direction. This was sufficiently reminiscent of the blow up to motivate some exploration.

Geometrically, to blow up a smooth point p in a surface X, replace p by the set of tangent lines to X at p. In a system of local coordinates centered at p, tangent lines are parameterized by slope or angle, with the caveat that a vertical line with slope ∞ or angle $\pi/2$ is the same vertical line as the one with slope $-\infty$ or angle $-\pi/2$. The geometric object that provides us with this directional coordinate is called the projective line, and denoted $\mathbb{P}^1$. It can be challenging to imagine this replacement of a zero-dimensional object (the point p) with a one dimensional thing (a projective line), but thinking topologically can help. Instead of replacing just the point p, we can replace a small disk centered at p. The lines through p are the diameters of this disk. When we blow up p, we separate these diameters so that they no longer all contain the same midpoint—there is a new version of the point p for each diameter, parameterized by slopes from $\mathbb{P}^1$. Since $\mathbb{P}^1$ is topologically a circle (for example, it can be described as the interval $[-\pi/2, \pi/2]$ with the endpoints identified), we have a circle's worth of diameters, which happen to be line segments. There are two options for a circle's worth of line segments: a cylinder or a Möbius strip.

A disk with a point blown up is a Möbius strip. Describe the disk centered at p as a unit disk in polar coordinates r and θ, and consider the following rectangle in the plane where (r, θ) are rectangular coordinates: $-1 \leq r \leq 1$, $-\pi/2 \leq \theta \leq \pi/2$, as in Figure 2. The fact that the limit of lines with positive slopes is the same vertical line that is the limit of lines with negative slopes tells us to identify the line $\theta = \pi/2$ with the line $\theta = -\pi/2$ via the identification $(r, \pi/2) \sim (-r, -\pi/2)$. In other words, identify the top and bottom of this rectangle using the half-twist $r \mapsto -r$, creating a Möbius strip. The classic reference [1] has more details for the reader interested in the algebraic geometry of the blow up, including a very nice drawing on page 29.

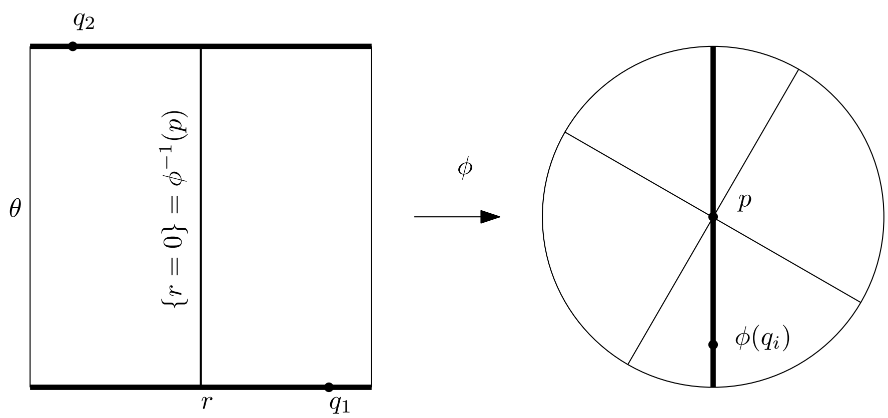

Figure 2: *Blowing up the point p. Observe that $\phi(0, \theta) = p$, and $\phi(r, \pi/2) = \phi(-r, -\pi/2)$*

The Shape

Latitude and longitude describe two families of circles on the surface of a sphere—the grid lines on a flat rectangular map. To make these lines into circles, the sides of the map (often somewhere in the Pacific) are identified, and the top and bottom of the rectangle are each collapsed to a point (the poles). As an intermediate step in this collapsing, it makes sense to connect each line of longitude, which is half of a great circle, to the line of longitude that is the other half of the same great circle. In other words, geodesics continue through the poles—one can fly over the north pole on the way from Beijing to New York and not notice that one has left the meridian $116°E$ and is now following $74°W$.

The Earth, like any rotating sphere, has two poles, where longitude becomes a direction instead of a location, so let us consider the shape generated by blowing up the north and south poles of a globe, and thus separating the great circles that pass through these two points. This gives us a circle's worth of circles in the same way that blowing up a point in a disk gave us a circle's worth of line segments. Similarly, there are two options for a circle's worth of circles: a torus or a Klein bottle.

A sphere with two points blown up is a Klein bottle. One way to see this is that a sphere with one point blown up is non-orientable, since the Möbius strip that is glued in by the blow up is not orientable, and blowing up another point doesn't change that fact. Thus to the extent that our sculpture represents a sphere with two points blown up, it is an illustration of the geometry of a Klein bottle.

We generated a family of circles by rotating a circle and simultaneously translating along the axis of rotation. After rotating a half turn, we arrive at a circle that we identify with the initial circle, so that the axis of rotation that we translate along becomes a circle parameterizing the circles in our family. The identification is visible in Figure 4b. The choice to make the translation along the axis of rotation equal exactly one diameter for one half turn was an aesthetic decision—we like how the resulting shape looks and feels. It does mean that at the center of the sculpture, there is a single point that represents both the north and south poles.

The surface can be parameterized by equations similar to the typical equations parameterizing a sphere in terms of a longitudinal coordinate θ and a latitudinal coordinate ϕ, but with an extra $\cos\theta$ added to the z coordinate to accommodate the blowing up.

$$x = \cos\phi\cos\theta \quad y = \cos\phi\sin\theta \quad z = \sin\phi + \cos\theta, \qquad 0 \le \theta \le \pi \quad -\pi \le \phi \le \pi.$$

The circles created by holding θ constant are shown in Figure 3.

x-axis *y-axis* *z-axis*

Figure 3: *Three views of the sculpture during construction.*

Adding a texture to the surface of our sculpture helped to illustrate the non-orientability of a Klein bottle. Away from the poles, the curvature of the longitudinal circles indicates a side of the surface that corresponds to the outer surface of the sphere. However, at the poles, this notion of outside has a discontinuity. In Figure 4b the part of the map showing Asia is stretched over the steel circles, but across the Arctic Ocean, Greenland and North America are visible as if from below.

Another interesting feature of the surface that relates to this discontinuity in the notion of the outside of the surface is that since the Klein bottle has Euler characteristic zero, integrating the Gaussian curvature of the shape gives zero. By inspection, there are regions of positive and negative curvature. For example, the helical equator traverses an area of positive curvature, while the axis of translation—where the poles have been blown up—is in an area of negative curvature.

Construction

The sculpture consists of an armature of stainless steel rods and a covering of latex sheet. We bent the rod into nine circles approximately a foot in diameter. These rings we attached to a central rod for the translation axis. Each ring is welded to the axis at two opposite points and the angles vary evenly, so that the nine circles match up with lines of longitude separated by 45° (the first and last rings represent the same great circle). We also made a helical rod for the equator. The rods were TIG welded at the attachment points. Two rectangles of latex were used to sheet the sculpture, one for each hemisphere created by the great circle of 0° and 180° longitude. The latex covering was stretched over this structure and held in place by gluing it to itself. The sides of each rectangle wrap around the rings corresponding to 0° and 180° longitude, and the top and bottom glue of each to the other rectangle across the axis of blown up poles. We used a sharpie to draw freehand the outlines of the continents and large islands to indicate the underlying sphere.

(a) *Lun-Yi adding the sheeting.* **(b)** *The 0° longitude line passes through west Africa.*

Figure 4: *Latex sheeting was stretched and glued to the steel structure, and a map drawn on the surface.*

At first glance, the surface represented in the sculpture appears to have a boundary, in fact two boundary components, but inspection of the outlines drawn on the latex indicates that this boundary is the circle consisting of the lines of longitude at 0° and 180°. So the sculpture indicates the desired identification by means of the map. For example, the coastline of Africa is visible on both sides of the apparent boundary component in Figure 4b.

Unfortunately, the latex sheeting deteriorated within a couple years of the construction. The shape was interesting enough that Lun-Yi made another version with steel sheeting on one hemisphere. This created a new set of challenges, as the latex naturally stretched to accommodate the changing curvature of this surface, but the steel had to be shaped. A new covering for the original piece will be created for Bridges 2018.

Acknowledgements

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References

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