Card Shuffling Visualizations

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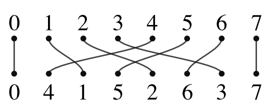
Abstract

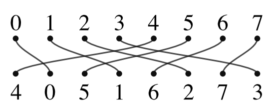
This paper discusses a few natural ways to visualize card shuffles, like the *perfect in- and out-shuffle* and the *milk shuffle*. These visualizations are designed to highlight interesting properties of the shuffles, like the *order* of the shuffles and the *stay-stack principle*. Such properties have attracted magicians and mathematicians alike, and visualizations like these make it possible to see properties that would have been hard to see otherwise.

Four Ways to Shuffle a Deck of Cards

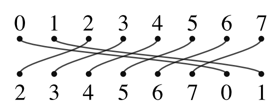
Imagine that you have a deck of cards, say eight cards numbered from 0 to 7: [0, 1, 2, 3, 4, 5, 6, 7]. Split the deck precisely into two halves, [0, 1, 2, 3] and [4, 5, 6, 7], and then interleave the cards evenly such that the top card remains at the top and the bottom card remains at the bottom. The result is [0, 4, 1, 5, 2, 6, 3, 7]. This is called a *perfect out-shuffle*. The corresponding *perfect in-shuffle* is the interleaving of the two halves such that the top card becomes the *second* card from the top: [4, 0, 5, 1, 6, 2, 7, 3]. (Perfect shuffles are also referred to as *faro* shuffles.) Another way to shuffle a deck is to give it a *cut*. This is where the top part of the deck, say [0, 1], is placed at the bottom, giving us [2, 3, 4, 5, 6, 7, 0, 1]. Finally, we have the *milk shuffle*. This is where we repeatedly pinch the deck and pull off the top and the bottom card into a separate pile. This move, often referred to as milking the deck, is repeated until all the cards have been pulled off. In this case, the first pair of cards to be pulled off is [0, 7]. Then, [1, 6], [2, 5], and finally [3, 4], which becomes the two top cards: [3, 4, 2, 5, 1, 6, 0, 7]. A nice and clean way to visualize these shuffles is with the following type of diagrams:



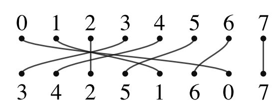
perfect out-shuffle



perfect in-shuffle



two-card cut



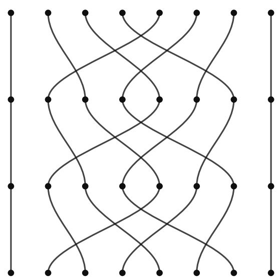
milk shuffle

The top rows represent the cards in a deck *before* each shuffle, and the bottom rows represent the order *after* each shuffle. Observe for example that in a perfect out-shuffle, the top card remains on the top, while in a perfect in-shuffle, the top card becomes the second card. The diagrams may also be read from *below*, or – equivalently – turned upside-down, in which case the shuffles are called the respective *reverse* shuffles. A reverse perfect shuffle occurs naturally when cards are being dealt or separated into distinct piles.

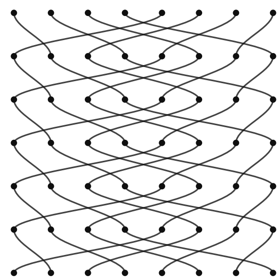
Permutations and the Order of Card Shuffles

From a mathematical point of view each shuffle is a *permutation* of the cards, and repeated shuffling can be viewed as the *composition* of permutations. Visually, this corresponds to stacking diagrams like this on top of each other. In more mathematical terms, the perfect out-shuffle on a deck of N cards, where N is even, moves the card in position p to the position $\mathbf{O}(p) \equiv 2p \pmod{N-1}$ for all p less than $N-1$ and to $\mathbf{O}(N-1) = N-1$ when p is $N-1$. In a similar way, the perfect in-shuffle moves the card in position p to $\mathbf{I}(p) \equiv 2p+1 \pmod{N+1}$. A cut of c cards simply moves the card in position p to $\mathbf{C}(p) \equiv p - c \pmod{N}$.

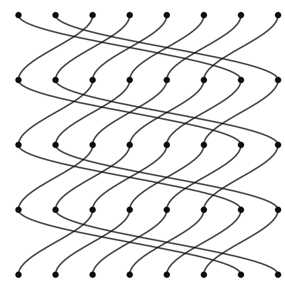
We will now consider what happens as we *repeat* the various shuffles. After a certain number of repetitions, called the *order* of each shuffle, the deck will return to its original order. The order of a particular shuffle depends on the number of cards: For eight cards, this happens after three times for the out-shuffle, six times for the in-shuffle, four times for the two-card cut, and four times for the milk shuffle:



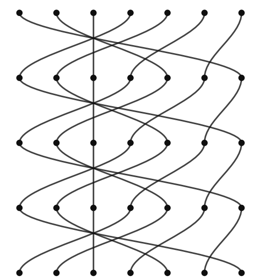
3 *out-shuffles with* 8 *cards*



6 *in-shuffles with* 8 *cards*



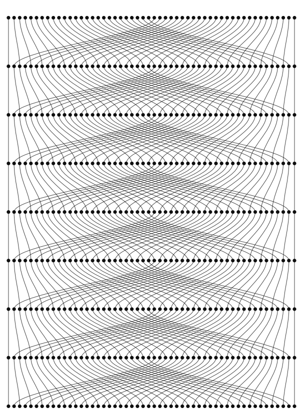
4 *two-card cuts with* 8 *cards*



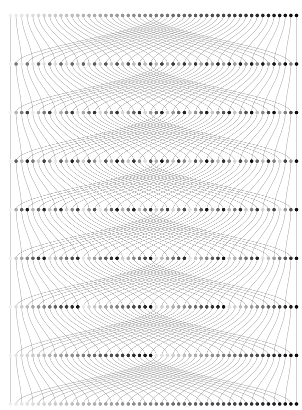
4 *milk shuffles with* 8 *cards*

Choices for Visualizations

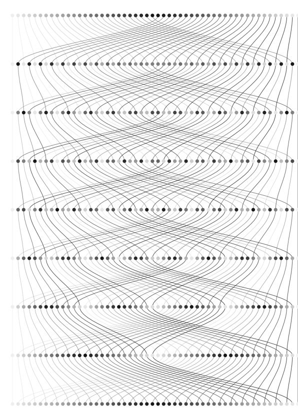
While pleasing to look at in their own right, we can do much more in terms of visualizing the underlying structures of each shuffle. Two obvious things we can do is to color the dots and the curves in various ways. There are numerous possibilities for the size, shape, and color of the dots and curves, and these are just a few:



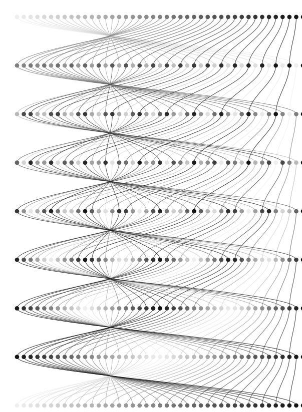
(a)
8 *out-shuffles with* 52 *cards*
dots: black
curves: black



(b)
8 *out-shuffles with* 52 *cards*
dots: light-to-dark
curves: faded



(c)
8 *out-shuffles with* 52 *cards*
dots: light-dark-light
curves: light-dark

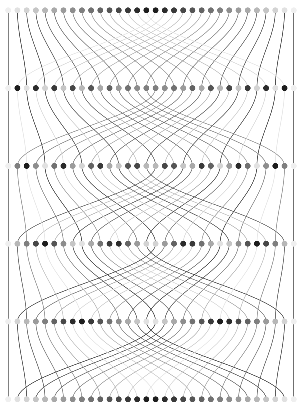


(d)
8 *milk shuffles with* 43 *cards*
dots: light-dark
curves: light-dark

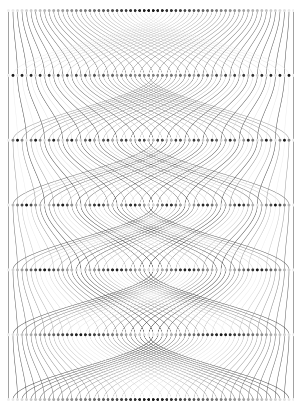
In all of these examples, the original order of the cards is restored after eight shuffles. In (a), (b) and (c), the shuffles are all perfect out-shuffles with 52 cards. For comparison, the shuffle in (d) is a milk shuffle with 43 cards. In (a), there is no particular coloring; consequently, all we see is an identical permutation being repeated eight times. In (b), the dots are colored from light to dark, while the lines are faded to give emphasis to the dots. The coloring of the dots makes it easier to see what the perfect out-shuffle actually does to the cards. For example, we can easily see the intermediate orderings and that the original ordering has been restored after eight times. In (c) the dots are colored from light to dark and *back again*, which is motivated in the next section, while the curves are colored from light to dark. In this way, it becomes apparent that the cards are being *weaved* together. In (d), both the dots and the curves are colored from light to dark.

Visualizing the Stay-Stack Principle

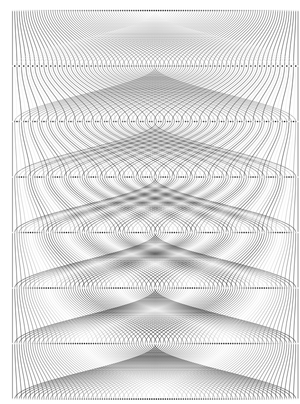
There are many interesting mathematical facts about card shuffling. One is the so-called *stay-stack* principle: If a deck of cards is arranged such that each card n from the top is *matching* the corresponding card n from the bottom, then the deck is said to have *central symmetry*. The stay-stack principle is that any number of perfect in- or out-shuffles preserves this central symmetry. The following three diagrams illustrate repeated perfect out-shuffles until the original order of the deck is restored. The dots are colored from light to dark and back again. Each curve is colored as the opposite of its starting point. Observe that all the diagrams have a left-right symmetry due to the initial coloring and the stay-stack principle.



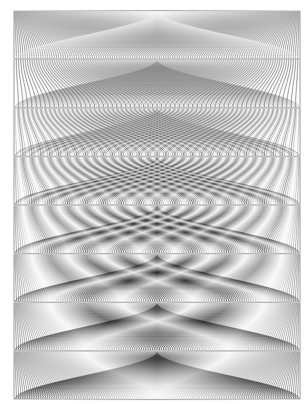
5 *out-shuffles with* 32 *cards*



6 *out-shuffles with* 64 *cards*



7 *out-shuffles with* 128 *cards*



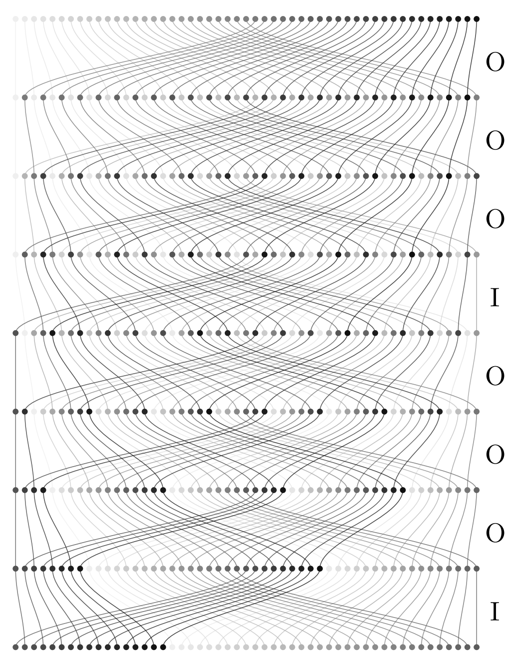
8 *out-shuffles with* 256 *cards*

Forcing a Card to the Top

It is now a well-established fact (see [3] for details) that any card in any deck can be forced to the top by an appropriate combination of perfect in- and out-shuffles. The history of this problem goes back to the magician and computer programmer, Alex Elmsley, who in 1957 published a method for forcing the *top card* to any position in a deck. See [1] for a fun typographical application of this principle.

Let us here illustrate a related property of perfect shuffles, which has a similar feel to it: In a deck of N cards, where N is odd, if $2^k \equiv 1 \pmod N$, then *any* sequence of k perfect in- and out-shuffles corresponds to *cutting the deck* somewhere. (For details, see [4].)

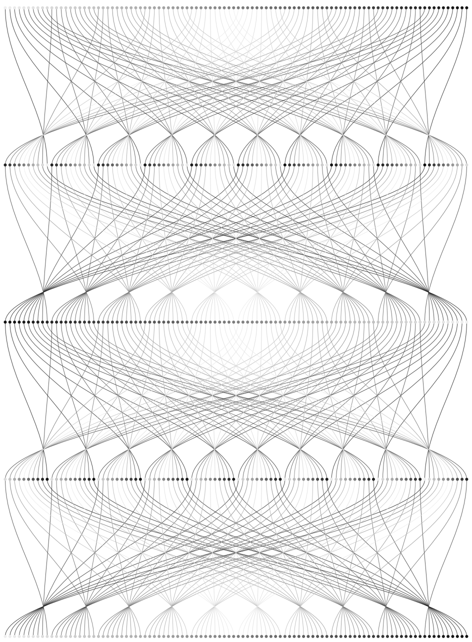
For example, in a deck of 51 cards, any combination of *eight* perfect in- and out-shuffles corresponds to a cut. The calculation checks out: $2^8 = 256 \equiv 1 \pmod{51}$. In the illustration to the right, the sequence *out-out-out-in-out-out-out-in* of perfect shuffles leads to a cut of 17 cards from the bottom of the deck. It is no coincidence that $17_{10} = 00010001_2$. A different combination of in- and out-shuffles will lead to a different cut. In this example, the result is that card 35 ends up at the top of the deck.



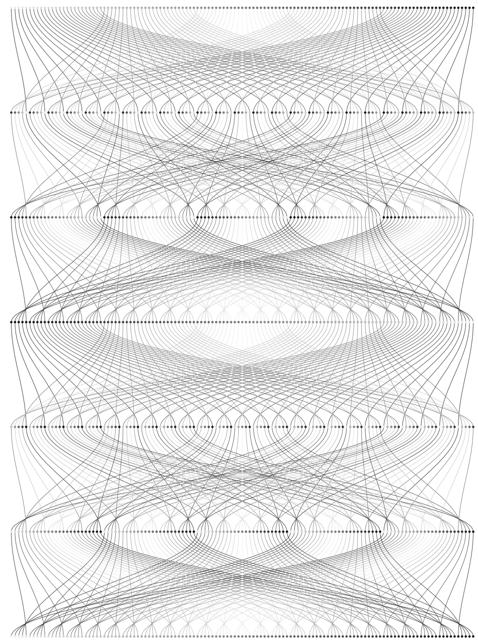
8 *perfect out-and in-shuffles*
forcing card 35 *to the top*

Generalizations

Perfect in- and out-shuffles, as well as several other shuffles, may be generalized to any number of cards and any number of piles. Here are two examples that I like, with 10 and 5 piles, respectively:



4 *in-shuffles with* 100 *cards and* 10 *piles*



6 *in-shuffles with* 125 *cards and* 5 *piles*

Conclusions and Acknowledgments

The inspiration for this work came from attending a talk by Perci Diaconis [2] last year. I wanted to know if I could *understand* the mathematics of card shuffling better through appropriate visualizations, so I programmed a sketch in Processing [5]. My motivation was to *see* some of the underlying structure that would be hard to see otherwise, much like with fractals and other computer-generated art. I find the process of visualizing and experimenting with mathematical structures through computer code, rewarding and exciting. These permutation diagrams are nice examples of how code can be utilized in order to visualize, and make tangible, mathematical structures, and also uncover some of that underlying beauty and complexity that often follow from very simple assumptions, in this case specific permutations corresponding to card shuffles.

References

[1] E. D. Demaine and M. L. Demaine. "Juggling and Card Shuffling Meet Mathematical Fonts." in *Connections in Discrete Mathematics: In Honor of Ron Graham's 80th Birthday*. Cambridge University Press, to appear.

[2] P. Diaconis. "The Magic of Charles S. Peirce." Mathematical Sciences Research Institute. April 2017.

[3] P. Diaconis and R. Graham. "The Solutions to Elmsley's Problem." *Math Horizons*, vol. 14, no. 3, 2007, pp. 22–27.

[4] S. Morris. *Magic Tricks, Card Shuffling and Dynamic Computer Memories*. ser. MAA Spectrum. Mathematical Association of America, 1998.

[5] Processing. https://processing.org/.