Just Intonation Keyboard: Isomorphic Keyboard Reimagined

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Abstract

The construction of a musical user interface providing access to all pure harmonies is a very complex problem that defies a completely satisfying solution. The notion of the so-called isomorphic keyboard, invented by Bosanquet and advocated by many key figures in tuning theory since the 19th century, offers a good but not a perfect solution. In this paper, I propose an alternative approach based on the concept of a 'just intonation (JI) plane.' A JI plane is a true isomorphic image of higher-dimensional affine spaces modeling tones in extended just intonation. JI keyboards are defined as sets of keys located in a JI plane.

Introduction

Since ancient times, musicians have preferred systems of tones based on simple frequency ratios. Two tones with frequency ratios such as $2/1$ (octave), $3/2$ (perfect fifth), $5/4$ (just major third), and $7/4$ (natural seventh) are perceived as pleasant and pure. At the same time, it is highly preferable that a system of tones can be accessed through a user interface with a regular structure. Musical keyboards should be such user interfaces, and their regular structure is often described through the notion of *isomorphism*, in which equivalent spatial patterns on a keyboard should result in equivalent patterns in musical pitch. The purity of harmonies and the regularity of the structure seem to be two crucial principles guiding the construction of musical keyboards.

The layout of a standard piano keyboard tuned in the usual 12-tone equal temperament adheres to neither of the two principles. First, several intervals are out of tune, some of them significantly. Strictly speaking, no interval is tuned purely, except for the octaves. All other intervals are only approximated. For instance, the pitch height of a perfect fifth is off by 2 cents and that of a major third is off by 14 cents.[1] While 2 cents might seem negligible, the mistuning of the major thirds is clearly noticeable. What is more, several just intervals are missing from the 12-tone equal temperament and cannot be sensibly approximated. The absence of a natural seventh ($7/4$) and a natural augmented fourth ($11/8$) is especially conspicuous. Second, the piano keyboard is not *isomorphic*: equivalent spatial patterns on the keyboard do not result in equivalent patterns in musical pitch. The same key formations give different musical chords (e.g., the major triad c–e–g and the minor triad d–f–a are sounded in the same way) and different key formations may give the same musical chords (e.g., major chords are performed in 6 different ways).

These issues have been addressed by musicians and theorists for a long time. Vicentino's [9] *archicembalo*, Bosanquet's [1] harmonium, Oettingen's [5] *orthotonophonium*, and Fokker's [3] 31-tone organ are just some of the many historical examples of constructions addressing the shortcomings mentioned above. Bosanquet's solution is special because it is the first that explicitly addresses the issue of isomorphism. Most modern designs of alternative keyboards followed Bosanquet's idea [3, 8, 10, 6]. Most recently, Milne, Sethares, and Plamondon [4] proposed a robust generalization of such keyboards.

Although Bosanquetian keyboards are highly regular, they usually compromise the principle of just intonation (JI). They typically assume an equal temperament of higher cardinalities, such as 31, 53, or 72. The model proposed in this paper enables the construction of isomorphic keyboards with strict adherence to the principle of JI. In this regard, it may be considered an improvement and a generalization of Poole's [7] *enharmonic keyboard*.

[1]The interval of an octave comprises 1,200 cents. Thus, the equal-tempered semitone measures 100 cents.

Mathematical Framework

Let $2 = p_1 < \cdots < p_n$ be the first n prime numbers and $q_1, \ldots, q_n$ be rational numbers. We say that the term $p_1^{q_1} \ldots p_n^{q_n}$ is *the prime number representation* of *a divisional tone* in the *p_n-limit system*. Prime number representations are unique up to trivial coefficients $p_i^0 = 1$. Furthermore, we say that a tone is *just* in the p_n-limit system if all superscripts $q_1, \ldots, q_n$ in its prime number representation are integers. Finally, just tones in the 3-limit system are called *Pythagorean*. The concept of divisional tones describes tones in any extended just intonation, any equal temperament, and any meantone tuning within a single model.

The divisional tones of a p_n-limit system can be embedded in an n-dimensional Euclidean space E_n where axes correspond to the prime numbers and the coordinates to the superscripts of their prime number representations. We call this model the *p_n-limit space*. Let us denote D_n its affine subspace (over the field of rational numbers) given by the points corresponding to the divisional tones. Finally, the subset of just tones corresponds to an n-dimensional lattice in this model.

We define *the pitch height* as a linear function on E_n by the formula $\mathbf{h}(q_1, \ldots, q_n) = 1200 \log_2(p_1^{q_1} \ldots p_n^{q_n})$. The pitch height of a tone is defined by the pitch height of the corresponding point in E_n. Obviously, the pitch height is defined for all points of E_n and not only for the divisional tones. Points with the same pitch height comprise parallel hyperplanes in E_n. We call them *isotone hyperplanes*. The line perpendicular to the isotone hyperplanes and containing the origin of E_n is called *the pitch-height axis*.

Let us observe the following crucial implication of the uniqueness of the prime number representation of divisional tones. While infinite subsets of E_n may have an equal pitch height, distinct divisional tones always have distinct pitch heights. Thus, the pitch height uniquely defines the divisional tones and the restriction $\mathbf{h}|_{\mathrm{D}_n}$ is a bijection.

We are ready to define our key concepts. Assume a two-dimensional plane P that contains the pitch height axis and an affine mapping μ of E_n onto P that preserves the pitch height function, i.e., $\mathbf{h}(a) = \mathbf{h}(\mu(a))$ for all $a \in P$. We call the mapping μ a *JI morphism* and the plane P a *JI plane* in the p_n-limit system.[2] Because of the uniqueness of pitch height values, as summarized in the previous paragraph, the restriction $\mu|_{\mathrm{D}_n}$ of a JI morphism μ to the subspace of divisional tones D_n is an affine isomorphism. This result is the most crucial mathematical finding of this paper.

Under a JI morphism, the images of the isotone hyperplanes are perpendicular to the pitch height axis and we call them *isotones*. The isotone that contains the origin is called *the pitch width axis*. The coordinate system given by the pitch height and pitch width axes plays a crucial role. The coordinate values on the pitch width axis define the *pitch width* as a linear function $\mathbf{w}$ acting on E_n.[3] A pitch width function is called *trivial* if the resulting JI morphism maps the E_n onto a line.

Finally, assume a JI morphism μ, its corresponding JI plane P, and a non-trivial subset K of divisional tones D_n. Then, the set of tones K, the image $\mu[K] \subset P$ of K under μ, and the planar points $\mu(k)$ for $k \in K$ are called *key tones*, *JI keyboard*, and *keys*, respectively. If additional pitch values approximating the actual pitch heights are attached to the keys of a JI keyboard, we speak of a *retuned keyboard*. Thus, a retuned keyboard is a set $\{(\mu(k), \mathbf{h}'(k)),\ k \in K\}$ where the real numbers $\mathbf{h}'(k)$ approximate the pitch height values of the key tones $\mathbf{h}(k)$. All Bosanquatian keyboards mentioned above [1, 3, 8, 4] are retuned keyboards in the 3-limit system where the Pythagorean tones are the key tones and the pitch height values are approximated by certain equal temperaments. Erv Wilson's mappings of 'constant structures' on the hexagonal keyboards are retuned keyboards in the 3-limit system, as well.

[2] Although a JI morphism is assumed to act on the entire space E_n and a JI plane contains an entire image of it, and thus they are not formally restricted to its subset of just tones, we still use the qualifier 'JI' to emphasize that our theoretical framework is directed toward a holistic understanding of just intonation.

[3] My concept of pitch width is directly inspired by the homonymous concept introduced by Clampitt and Noll [2] and is a generalization thereof.

Basic Examples of a JI Keyboards

A JI keyboard is a set of keys located on a JI plane. The JI plane has two axes: the pitch height and the pitch width. We may assume that the pitch height runs in the horizontal direction and the pitch width in the vertical direction. The divisional tones, as a superset of just tones, are isomorphically embedded in the JI plane: equivalent spatial patterns on the JI plane correspond to equivalent pitch patterns and vice versa.

A set of n divisional tones is called a *JI basis* if the set of the n corresponding points in the affine space E_n along with the origin determines an affine basis of E_n. A JI morphism is fully determined by the pitch width values for a JI basis and any choice of pitch width values for a JI basis uniquely determines a JI morphism. Thus, the construction of a JI plane may comprise two steps: (1) choosing a JI basis and (2) defining the pitch width values for the JI basis.

The standard basis of E_n corresponds to the series of first n prime numbers, which is thus a straightforward choice of a JI basis. Sometimes, though, considering a different basis may be more convenient. For instance, if our concern was to adjust the position of just tones of the 13-limit system within an octave, we could consider the basis $\{2, 5/4, 11/4, 3/2, 13/8, 7/4\}$. Alternatively, we may want to derive the just tones of the 11-limit system from the Pythagorean tones and a system of small intervals called commas. Then, we may consider a basis such as $\{2, 3/2, 81/80, 63/64, 33/32\}$. In our example we assume this JI basis.

The width function defines the vertical location of tones on a JI plane. We call the function $\mathbf{w}_s(p) = -1$ for all prime numbers *p the standard pitch width*. Many different non-trivial width functions can be constructed to achieve the particular structures of JI keyboards. In our example, we consider a slight modification of the standard pitch width given by the following mapping: $2 \mapsto -1$, $3/2 \mapsto 0$, $81/80 \mapsto 1$, $63/64 \mapsto 1.25$, and $33/32 \mapsto 3$. In this case, for instance, the widths of the Pythagorean whole tone $9/8$, diatonic semitone $256/243$, and chromatic semitone $2187/2048$ are 1, -3, and 4, respectively.

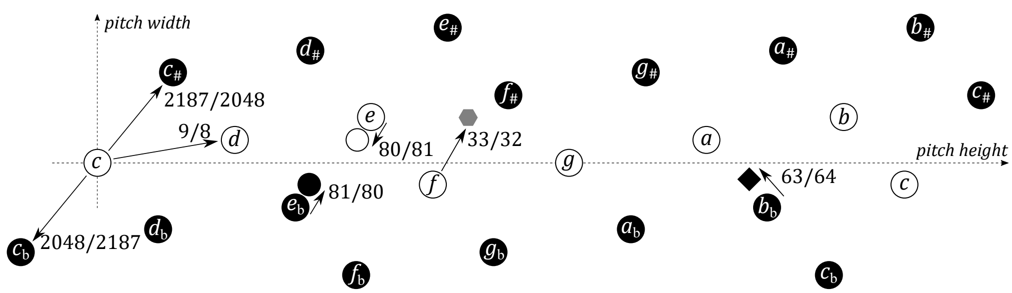

Figure 1: *A simple example of a JI keyboard.*

Figure 1 shows a simple JI keyboard. The unlabeled round keys with usual letter labels represent the Pythagorean tones. For the letter labels, we assume that 1 corresponds to the tone c. The smaller round white key shows the major third in JI, i.e., the tone $5/4$, which may be derived from the Pythagorean major third $81/64$ by the correction of syntonic comma $80/81$. Similarly, the unlabeled round black key shows the minor third in JI, which corresponds to the Pythagorean minor third corrected by the syntonic comma in the opposite direction. The black diamond marks the natural minor seventh $7/4$, derived from the Pythagorean minor seventh through the correction of the septimal comma of $63/64$. Finally, the natural augmented fourth $11/8$ of the 11-limit system is shown by the gray hexagon. It may be achieved by correcting the Pythagorean fourth by the undecimal comma of $33/32$.

Figure 2 shows a more complex example of a JI keyboard in the 11-limit system. The graphical representation of keys follows the convention explained in the previous example. The largest round keys mark the Pythagorean tones and the smaller round keys mark their syntonic comma corrections. Thus, the system of round keys provides a relatively large selection of just tones in the 5-limit system. The diamonds show the just tones in the 7-limit system and, finally, the hexagons represent just tones in the 11-limit system.

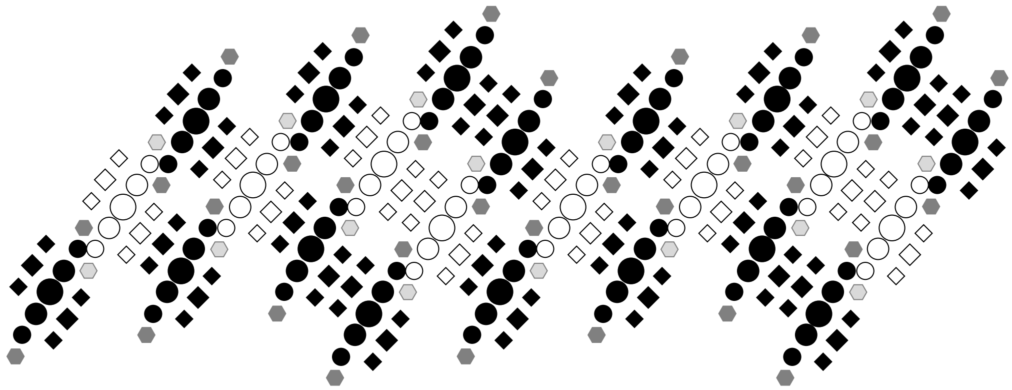

Figure 2: *A more complex JI keyboard in the 11-limit system.*

Conclusion

We proposed a theoretical framework that enables the construction of isomorphic musical keyboards providing access to a wide range of tones in extended just intonation. The key components of the framework are the concepts of divisional tones, JI morphism, JI plane, and pitch width. This paper summarizes the theoretical results of the preparatory phase of a larger project that aims to develop specialized musical software focusing on extended just intonation.

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