Rock Me Fibonacci: Using Recurrence Relations and State-Transition Matrices to Count Rock Drum Fill Patterns

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Abstract

Drum fills are a way of "filling time" during a short break in a rock or pop song, usually using a rapid sequence of notes played in succession across multiple drums in a drum kit. Considering the common configuration known as a five-piece kit, we see that moving from one drum to another is considerably easier in some combinations than others. We construct a set of rules to model patterns which avoid these difficult transitions, and then construct and solve recurrence relations to count the number of n-beat patterns which fit the rules. Variations on the theme are provided by varying the durations of notes and the numbers and types of drum allowed.

Drumming and Drum Fills

The connection between mathematics and rhythmic drumming has been recognized for a long time, but most mathematical attention has focused on rhythms played with one hand on one drum. Some exceptions include [2], which looks at multiple drum lines, and [5, Chaps. 29 and 32], which discusses drumming with both hands. As far as I know, however, there has been little mathematical study of how the spatial position of the hands affects the feasibility of drum patterns and in particular how this applies to rock drumming.

Rock drums are played on a drum kit, such as the one shown in Figure 1. This is a typical example of a "five-piece" kit used by a right-handed player. The five pieces refer to the snare drum, high tom, bass drum, mid tom, and low tom, as labeled in Figure 2. The three cymbals shown (high hat, crash, and ride) are also fairly typical.

Drum notation uses a standard musical staff and notes, with the lines and spaces repurposed to indicate the different drums and cymbals rather than actual pitches. One common system puts cymbals on the top line and above, bass drum on the lowest space, and the other four drums in between, as indicated by the abbreviations on the left side of Figure 3. It is also usual for the note heads to be replaced by $\times$ shapes for cymbals, as in the figure.

There are basically two different kinds of rock drum patterns: "playing time" and "filling". Playing time, as shown in Figure 3, involves a repeated pattern which provides a background for the other instruments and vocals. Filling, as shown in the last part of Figure 4, is a "mini-solo" for the drummer, usually occurring during a break in the melody carried by the lead singer or instrumentalist.

Counting

Our goal is to count the number of fills possible for a given number of beats under certain constraints. For the first part of this paper, these constraints will be:

1. Only the snare drum and the three toms are used in the fill, as shown for example in Figure 4.
2. The notes are all the same length, as again shown in Figure 4.

Figure 1: *Platin Drums Classic Set 2216 Amber Fade [Wikipedia user "Mark dolby"]*

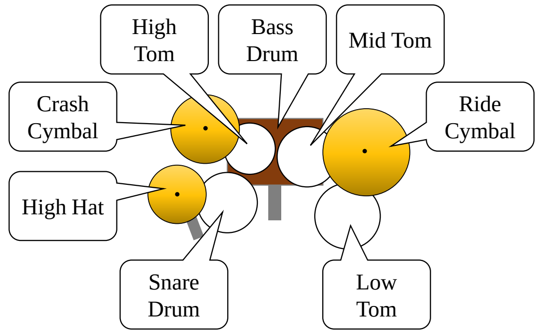

Figure 2: *Layout of a typical (right-handed) five-piece kit*

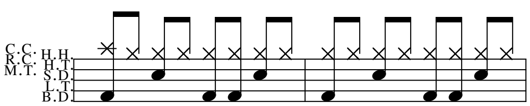

Figure 3: *From "Are You Gonna Go My Way", Lenny Kravitz*

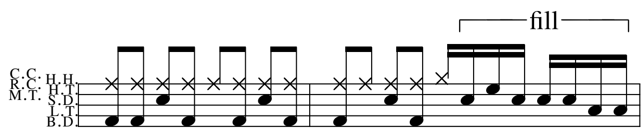

Figure 4: *From "Crosstown Traffic", Jimi Hendrix*

3. The notes are played one at a time and "hand-to-hand", that is, alternating left and right hands. For example, the fill in Figure 4 would generally be played starting with the left hand on the snare as left, right, left, right, left, right, left.
4. Certain "difficult" transitions will be avoided.

To clarify Constraint 4, note that in Figure 1 not all of the four drums we are using are equally far off the floor. In particular, note that the high and mid toms are at about the same height and the snare and low tom are at about the same height. (The use of "high" and "low" in the name of the toms indicates pitch rather than spatial placement.) This makes two sets of transitions unusually difficult, though not impossible: right hand on snare to left hand on low tom or left hand on low tom to right hand on snare, and similarly for right hand on high tom and left hand on mid tom. In each of these cases, playing quickly requires the player's arms to cross in a way that makes it difficult for the arms to avoid hitting each other.

In order to count drum fill patterns, we could treat them as a string of the letters S, H, M, and L, with certain combinations disallowed. For example, the fill in Figure 4 could be represented as SHSSSLL. For the moment, let us assume that (unlike in Figure 4) the first beat of the fill is played with the right hand. Then Constraint 4 says that substrings of SL and HM are not allowed with the substring starting on a odd-numbered (right-hand) beat, and LS and MH are not allowed with the substring starting on a even-numbered (left-hand) beat. Fill patterns starting on the left hand can be dealt with similarly.

Without Constraint 4, there are 4^k strings of length k. With the constraint, we can count the number of allowed strings using standard recurrence relation techniques, as explained for example in [1, Chap. 5]. If s_k, h_k, m_k, ℓ_k are the number of allowed k-beat fills with the begining of the fill starting with the right hand and ending on the snare, high tom, mid tom, low tom, respectively, then $s_1 = h_1 = m_1 = \ell_1 = 1$ and

$$\begin{array}{rlllll} s_{2n} & = s_{2n-1} & +h_{2n-1} & +m_{2n-1} & +\ell_{2n-1} \\ h_{2n} & = s_{2n-1} & +h_{2n-1} & +m_{2n-1} & +\ell_{2n-1} \\ m_{2n} & = s_{2n-1} & & +m_{2n-1} & +\ell_{2n-1} \\ \ell_{2n} & = & h_{2n-1} & +m_{2n-1} & +\ell_{2n-1} \end{array}$$

$$\begin{array}{rlllll} s_{2n+1} & = s_{2n} & +h_{2n} & +m_{2n} & \\ h_{2n+1} & = s_{2n} & +h_{2n} & & +\ell_{2n} \\ m_{2n+1} & = s_{2n} & +h_{2n} & +m_{2n} & +\ell_{2n} \\ \ell_{2n+1} & = s_{2n} & +h_{2n} & +m_{2n} & +\ell_{2n} \end{array}$$

Note that this is similar to the way counting certain tilings produces the Fibonacci numbers, as for example in [1, Chap. 7].

It is also common to represent these equations with matrices, as explained for example in [3, Sec. 5.6]. These are sometimes called state-transition matrices. If we let $\mathbf{x}_k = (s_k, h_k, m_k, \ell_k)^T$ we can write our equations as $\mathbf{x}_{2n} = R\,\mathbf{x}_{2n-1}$, $\mathbf{x}_{2n+1} = L\,\mathbf{x}_{2n}$, $\mathbf{x}_1 = (1, 1, 1, 1)^T$, where

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then the solution to the equations is $\mathbf{x}_{2n} = R(LR)^{n-1}\,\mathbf{x}_1$ and $\mathbf{x}_{2n+1} = (LR)^n\,\mathbf{x}_1$.

The total number of drum fills is the sum of the entries of $\mathbf{x}_{2n} = R(LR)^{n-1}\,\mathbf{x}_1$ for even length patterns, or of $\mathbf{x}_{2n+1} = (LR)^n\,\mathbf{x}_1$ for odd length patterns. If we let $t_k = s_k + h_k + m_k + \ell_k$ be the total number of fills of length k, then diagonalizing the matrix LR (as explained in for example [3, Sec. 5.6]) and summing gives us:

$$t_{2n} = \left(\tfrac{1}{2} + \tfrac{5\sqrt{17}}{34}\right)\left(\tfrac{13}{2} + \tfrac{3\sqrt{17}}{2}\right)^n + \left(\tfrac{1}{2} - \tfrac{5\sqrt{17}}{34}\right)\left(\tfrac{13}{2} - \tfrac{3\sqrt{17}}{2}\right)^n$$

$$t_{2n+1} = \left(2 + \tfrac{8\sqrt{17}}{17}\right)\left(\tfrac{13}{2} + \tfrac{3\sqrt{17}}{2}\right)^n + \left(2 - \tfrac{8\sqrt{17}}{17}\right)\left(\tfrac{13}{2} - \tfrac{3\sqrt{17}}{2}\right)^n$$

The first few values of t_k are 1, 4, 14, 50, 178, . . . , which is also sequence A055099 of the OEIS [4]. Using matrices to represent the recurrence relation is also one of the ways to derive the closed-form formula for the Fibonacci number, as shown for example in [1, Chap. 14]. That formula has a similar form to this one.

Variations on the Theme

We might also want to know whether we can repeat a drum fill pattern, as is done in Figure 5. For patterns of even length, the number of fills of length $2n$ which can be repeated is the same as the number of fills of length $2n + 1$ which start and end on the same drum. We can use the fact that each entry of $(LR)^n$ actually gives us the number of drum fills of length $2n + 1$ that start and end in specified places. Let r_k be the total number of fills of length k which can be repeated. The diagonal elements of the matrix $(LR)^n$ indicate patterns which start and end in the same place, and taking the sum of the diagonal elements (the trace) gives:

$$r_{2n} = 1 + \left(\tfrac{13}{2} + \tfrac{3\sqrt{17}}{2}\right)^n + \left(\tfrac{13}{2} - \tfrac{3\sqrt{17}}{2}\right)^n$$

For odd length patterns, the number of fills of length $2n - 1$ which can be repeated is the same as the number of fills of length $2n$ which start and end in the same place and can be played starting on either hand. These patterns can be counted in a similar way.

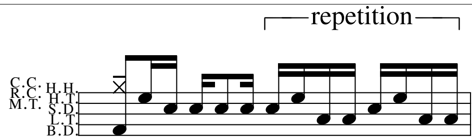

Figure 5: *From "Breakout", Foo Fighters*

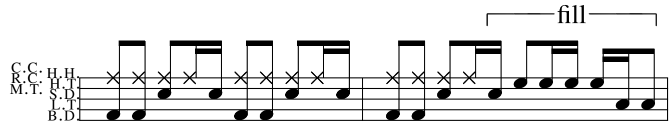

Figure 6: *From "Come Together", The Beatles*

Another thing we might want is to allow some notes to be twice as long as the others, as shown in Figure 6. (Note that if we play on just one drum, this is equivalent to a tiling which is counted by the Fibonacci numbers, as mentioned above.) One of the peculiarities of playing percussion is that playing a "long" note is in most cases equivalent to playing a note with a pause after it. Drummers often take advantage of this space to break the hand-to-hand rule, playing with the same hand both before and after the pause in order to make later parts of the pattern easier. On the other hand, in some cases it is easier not to break the rule. We can allow both cases by re-defining our state-transition matrix. We will now have only one matrix with nine rows and columns, one for each "state": right hand on each drum, left hand on each drum, and "pause". Schematically, this matrix looks like:

$$\left(\begin{array}{c|c|c} \mathbf{0} & R & \mathbf{1} \\ \hline L & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{1} & 0 \end{array}\right)$$

where **0** and **1** represent submatrices of the appropriate sizes filled with zeros and ones, respectively. The diagonalization and summation of this matrix is left as an exercise for the reader. Similar matrices could be used to introduce the bass drum pedal and other techniques that allow (or require) the drummer to break the hand-to-hand rule.

Summary and Conclusions

The state-transition matrix provides a powerful framework for counting all sorts of patterns that can be played on the drum kit. For example, we could add rows and columns representing more drums, cymbals, accents, flams, and simultaneous hits on more than one drum. With the appropriate care, more than two lengths, double- and triple-strokes, and rolls could also be accommodated, limited only by the size of the matrix one is willing to deal with.

One issue that we have not yet addressed is the fact that a subpattern which can be played by either breaking the hand-to-hand rule or not will be double-counted by our method. Dealing with this may require a more detailed examination of the number of ways patterns can be broken up into subpatterns. Considering the number of ways to parenthesize a product of variables might be useful here. This is related to the Catalan numbers, as shown in [1, Chap. 20].

References

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