Multi-Scale Truchet Patterns

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Abstract

In his paper on the pattern work of Truchet, Cyril Stanley Smith introduced a variant of Truchet's tile that gives patterns of continuous lines that delineate two-colorable domains. We generalize Smith's tiles to infinite sets of tiles scaled by powers of 1/2. Using the concept of a "winged" tile, we show an easy way of producing multi-scale patterns that show a large variety of emergent forms.

Multi-Scale Smith Tiles

At the end of his 1987 paper on Truchet's pattern tiles [7], Cyril Stanley Smith introduced a variant of Truchet's tile that has become the most popular expression of the concept. The motif on Smith's tile consists of two circular arcs that join the midpoints of adjacent sides (Figure 1, left). When the tiles are laid together in random orientations, the arcs connect to form continuous boundaries that separate closed domains. The domains can be colored alternately white and black to make infinitely variable patterns (Figure 1).

I wondered whether it is possible to devise a variant of Smith's tile that works at multiple scales. With a scale factor of 1/2, two scaled tiles adjoin to a side of one unit-scale tile. If Smith's two arcs are replaced by four arcs that meet the sides at 1/3 and 2/3, the arcs of the larger tile connect to arcs of the smaller tiles, leaving two loose ends in the smaller tiles (Figure 2).

In larger assemblies of these tiles, the loose ends accumulate, but fortunately are always even in number so that it is in theory possible to connect all the loose ends pairwise without ending up with an orphan. Indeed, with some patience and care, you can always find a way to complete the connections in a pattern without crossing lines to form continuous domains that can be colored black and white, as with Smith's tiles. The method described in the remainder of this paper is an informal constructive proof of those statements.

While completing the connections by hand is relatively straightforward, doing so algorithmically requires elaborate data structures and complex bookkeeping to keep track of the geometric and topologic relationships involved. Fortunately, there is a simpler and more direct way to produce multi-scale, two-color patterns that is accessible to anyone with rudimentary programming skills or access to a printer and scissors.

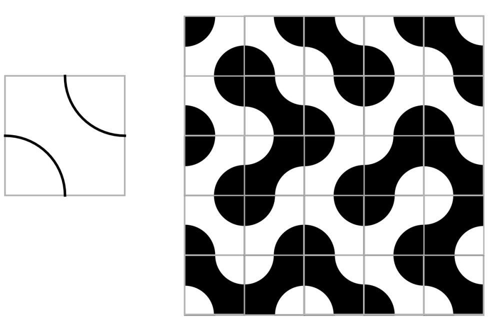

Figure 1: *Smith tile and colored pattern.*

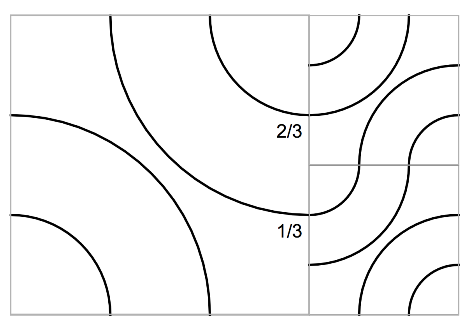

Figure 2: *Multi-scale tile connections.*

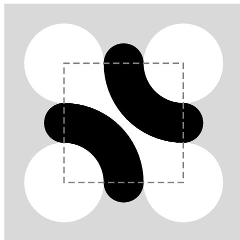

Figure 3: *Winged tile.*

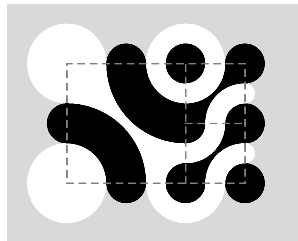

Figure 4: *A winged tile tiling.*

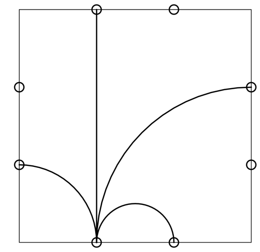

Figure 5: *Possible circular arc connections.*

Winged Tiles

The direct method uses the "winged" tiles illustrated in Figure 3. A winged tile consists of the content of the tile, shown in the figure within the square dashed boundary, plus "wings" that complete the motif outside of the boundary. The gray background is not part of the tile, but is included so that the irregular boundary of the tile is visible. The shapes of the wings are arbitrary, but for a given set of motifs there are usually obvious shapes that fit with the character of the patterns yielded by the motifs.

Winged tiles are assembled along their square content boundaries with the wings overlapping adjacent tiles. Using the motifs presented in this paper, the order of overlap makes no difference graphically with equal-scale tiles. When smaller-scale tiles are adjoined to larger ones, the *smaller tiles are always placed on top of the larger*.

One additional detail must be attended to to yield continuous black and white domains. *With each step of scaling, the colors in a tile are inverted*. Even parity tiles are black-on-white, and odd parity tiles white-on-black. When parity-colored tiles are assembled, their domains connect white-to-white and black-to-black, with the wings automatically completing the pattern (Figure 4). As you will see in the subsequent sections, the merging of tile domains and wings at multiple scales gives a rich variety of emergent forms not found in single-scale tilings.

Browne [2] gives another method of generalizing Smith tile contours to multiple scales. But note that these "multiresolution ST contours" differ from the winged tiles I describe both in method and result. Browne produces closed contours by tracing circular arcs through subdivision tilings, whereas winged tiles produce contours directly via overlaps.

Motifs

The eight connection points on the edges of multi-scale tiles admit many more ways of making internal connections than the four points on Smith's tiles. If we restrict attention to circular arcs and straight lines, each multi-scale edge point can be connected to four others (Figure 5). Taking all combinations of pairwise connections with each point connected to one other point yields the 24 possibilities shown in Figure 6.

Although any of those possibilities can be colored to yield a winged tile motif, in this paper I use only the fourteen tiles whose internal connections do not intersect. I add to those the "+" tile (Figure 6, first row, seventh tile), which graphically belongs in the set. The resulting winged tiles are shown in

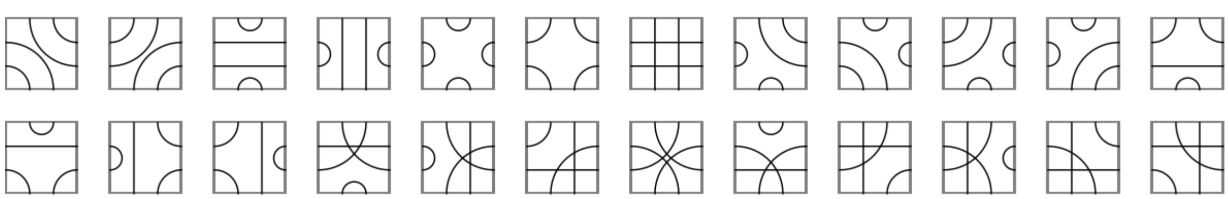

Figure 6: *The 24 distinct pairwise connections of edge points.*

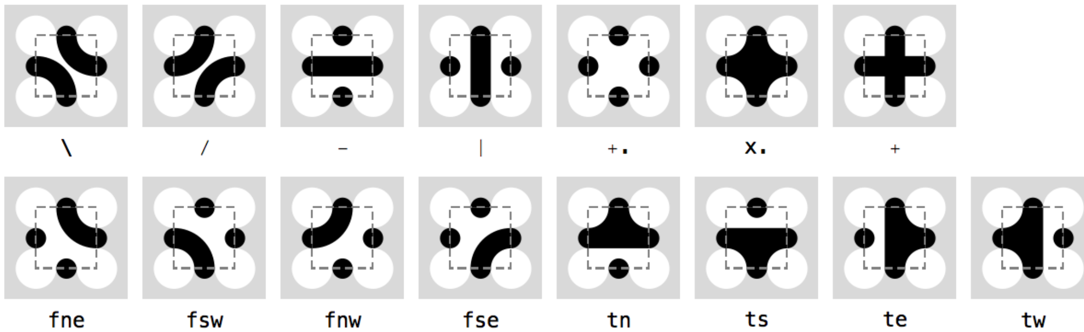

Figure 7: *The 15 winged tile motifs used in this paper.*

Figure 7, labeled with the ASCII short names that I use to refer to them ("fse" = "frown south east", "te" = "T east"). There are four groups of rotationally-equivalent tiles in the set, two with two members and two with four members. The number of distinct tiles considering rotational equivalence is seven.

Tilings

Multi-scale tiles can be joined in any configuration that corresponds to the relationship between a tile and a neighboring tile subdivided into smaller-scale tiles. That is to say that a 1/2-scale tile, for example, can be joined to a unit-scale tile at one of the two intervals [0,1/2], or [1/2,1] along a side (Figure 8).

Subdivision Tilings

Multi-scale tiles lend themselves naturally to subdivision tilings, where a tile pattern is derived by recursively subdividing a square into smaller squares. The subdivision process may be random or orderly (Figure 9).

I most often use random subdivision tilings to get a feel for the graphical qualities of various combinations of winged tile motifs. Even single-motif tilings give interesting graphics when the tiles are combined at multiple scales. Due to the parity-coloring of the tiles, multi-scale tilings display interesting foreground-background relationships, with patterns at different scales integrating naturally into a whole. A pattern at one scale is foreground for patterns at the next larger scale, and background for patterns at the next smaller scale. Figure 10 shows several examples of such single-tile patterns.

Combining multiple motifs in a tiling increases the variety of emergent forms it exhibits. Figure 11 shows examples of two- and three-motif tilings.

T-Uniform Tilings

A conventional "*k*-uniform" tiling has a finite number of tile shapes and *k* vertex types, with tile corners meeting only at vertices (see [3], Sec 2.2). I extend that term here to "*k*-T-uniform" tilings that include "T"-junctions, where two tile corners meet at the side of a third tile. T-uniform tilings decorated with one or more winged tile motifs give interesting wallpaper patterns. When the motifs are chosen randomly, the

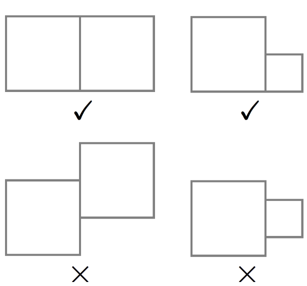

Figure 8: *Admissible and inadmissible tile relationships.*

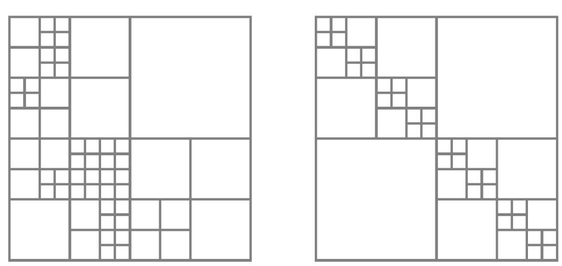

Figure 9: *Random and orderly subdivision tilings.*

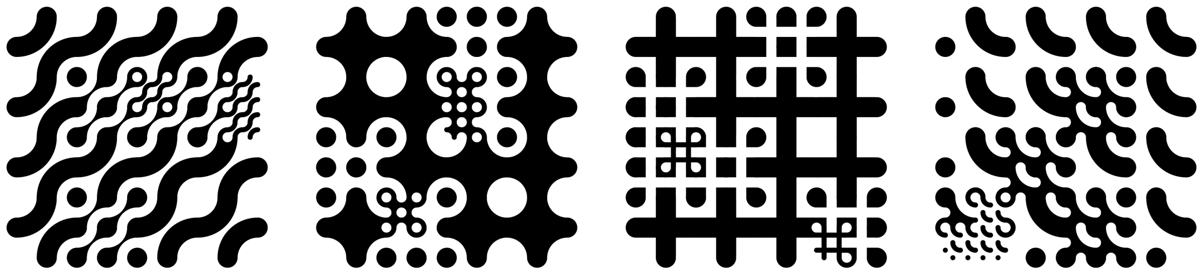

Figure 10: *Single-tile subdivision tilings: /, x., +, and fne motifs.*

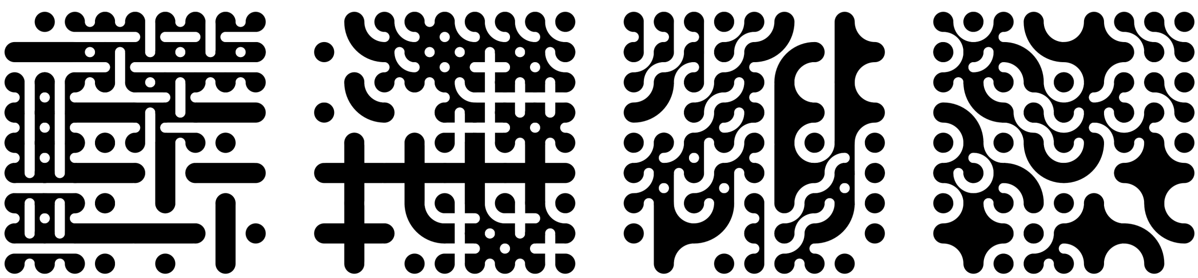

Figure 11: *Two- and three-motif subdivision tilings: (|, -), (+, +., fne), (/, fnw, te), and (\, fnw, x.).*

interaction between the uniform tiling and random motif pattern gives compelling patterns intermediate between regularity and irregularity.

Figure 12 shows four examples of T-uniform tilings. Figure 13 shows the same tilings decorated with various single motifs to give regular wallpaper patterns. Figure 14 shows those tilings decorated with various combinations of motifs chosen at random. While some global regularity in the patterns is apparent at first glance, closer inspection reveals a great variety of emergent forms within them.

Subdivision Fills

With tiles available at multiple scales, winged tiles can be used to fill arbitrary areas by recursive subdivision. A recursive fill tiling is produced by starting with a square that covers the region to be filled and recursively subdividing. At each subdivision step, sub-squares completely within the region are retained, sub-squares completely outside the region are discarded, and sub-squares that intersect the boundary of the region are further subdivided. The process continues to the desired resolution. Figure 15 shows the result of subdivision filling an annulus, and decorating the resulting tiling with \ and / motifs.

Because a winged tile extends beyond the square content area of the tile, the pattern that results from decorating a subdivision fill with winged tiles may show ugly unevenness where different-scale tiles lie along a boundary of the filled region. That undesirable effect is easily compensated for in the generation of the fill by triggering a subdivision when the bounding square of a winged tile rather than the tile

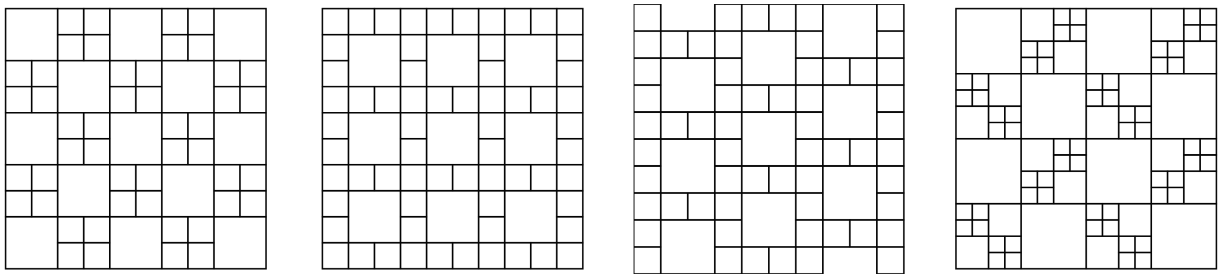

Figure 12: *T-uniform tilings.*

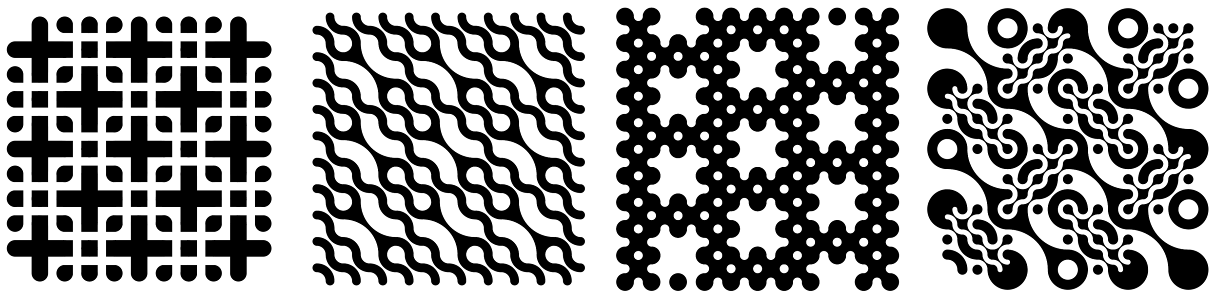

Figure 13: *Tilings of Figure 12 decorated with single motfs: +, \, +., and \.*

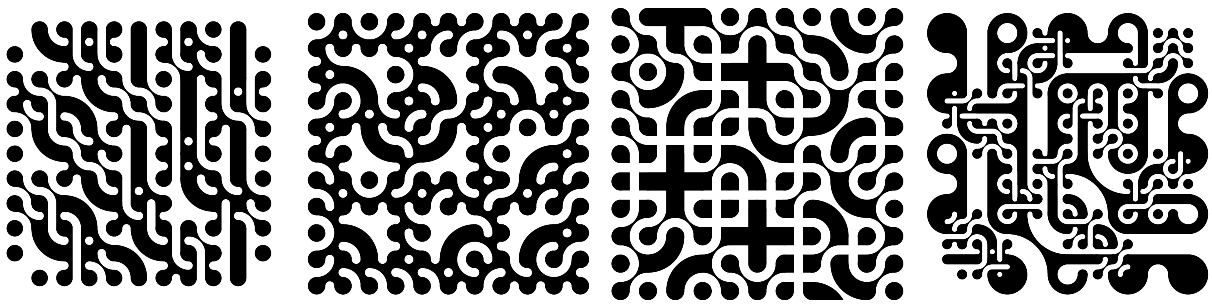

Figure 14: *Tilings of Figure 12 decorated with multiple random motifs: (|, \, fne), (fne, fnw, fse, fsw), (+, \, /), and (|, -, \, /).*

content square intersects the region boundary. Figure 16 shows the result of applying that correction, and Figure 17 the underlying subdivision fill.

Technical Considerations

For computer implementation, a tiling is most easily represented by a set of 3x3 homogeneous transformation matrices that specify the positions of the tiles. To render a tiling, the transforms are sorted by scale from large to small and then applied in that order to winged tile motifs to make a display list. Motifs are color-inverted or not depending on the parity of the transform that is applied to them. Since large-scale tiles precede smaller-scale tiles in the display list, small tiles are rendered on top of large ones.

A tile's scale is given by the square root of the determinant of its transformation matrix. The parity is calculated from the scale by looking at whether its base 2 log is even or odd.

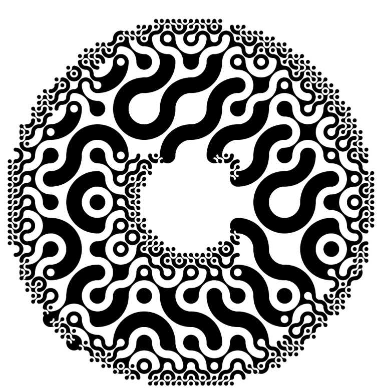

Figure 15: *Subdivision tiling, uncorrected.*

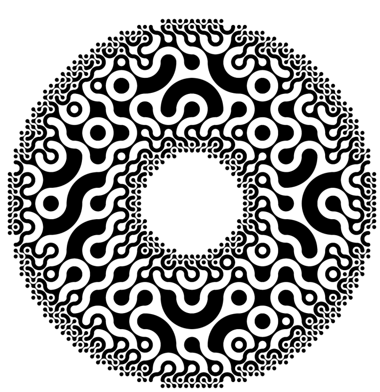

Figure 16: *Subdivision tiling, corrected.*

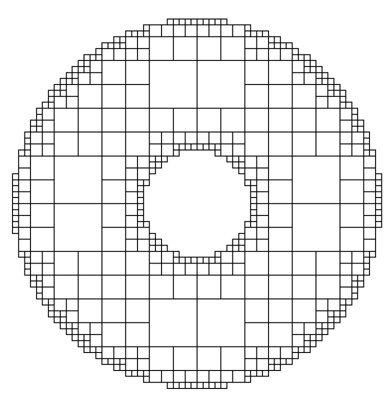

Figure 17: *Subdivision fill underlying Figure 16.*

I have implemented a prototype package for exploring multi-scale Truchet patterns in the Wolfram Language using Mathematica [6]. The package and numerous examples of its use are available for download at https://christophercarlson.com/portfolio/multi-scale-truchet-patterns/.

Conclusion and Future Directions

The winged tile concept makes multi-scale Truchet patterns accessible to a wide audience. The domain is rich in possibilities and ripe for exploration.

The concepts I have described here extend in the obvious way to triangular tiles. Perhaps they can be applied to non-regular rep-tiles as well. Mixtures of square and triangular tiles would extend the formal possibilities.

There are of course many more possible motifs than the ones composed of arcs and straight lines that I have described here. Krawzyck's review paper gives some possibilities worth exploring [4]. Borlenghi's blog post [1] gives an extensive exploration of Truchet motifs.

Whether scale factors other than 1/2 are possible merits investigation. Obvious candidates are size ratios of 1:3 and 1:4, but also 2:3 and 3:4, which would allow for more scaling steps before the details in a pattern become too small to be significant. Mixed ratios, for example 1:3 and 2:3 may also be possible.

By thinking in terms of adding and subtracting material rather than black and white colors, the winged tile concept can be extended to three dimensions. A 3D winged tile would consist of additive and subtractive volumes. When a tile is added to a tiling, its subtractive volume is removed from the tiling and its additive volume is added to it. 3D tiles offer the intriguing possibility of fascinating sculptural forms lurking in the emergent forms they would generate. But with the additional degree of freedom that 3D tilings have over 2D, it is paradoxically more difficult to design tilings whose surfaces connect smoothly and generate pleasing emergent forms because there are more parameters to coordinate. Lord and Ranganathan have suggested batwing tiles as a possibility [5], and Browne offers "multiresolution surfaces" [2]. It remains to be seen whether these or other concepts can be successfully adapted to 3D winged tiles.

References

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