Musical Scales and Multiplicative Groups

Donald Spector

Dept. of Physics, Hobart and William Smith Colleges, Geneva, NY, USA; spector@hws.edu

Abstract

Composers frequently apply mathematical operations to musical scales that treat a scale's sequence of notes in additive fashion. Here, we introduce operations from multiplicative group theory to develop new methods of obtaining and transforming musical expressions. The methods are applicable to conventional and non-conventional scales.

Introduction

It is well known that standard musical scales have a mathematical underpinning. In Western music, just intonation is formed by identifying frequencies in certain whole number ratios with particular musical intervals, while well-tempered scales are obtained by having successive notes have a frequency ratio of $2^{1/12}$, with octaves corresponding to a doubling of frequencies [4]. There are, of course, also microtonal [5] and non-octave scales [7].

In many contexts, however, it is customary to think of musical notes as forming an arithmetic sequence. Mathematically, one can think of this as labeling notes in terms of the logarithm of their frequencies, although that is not necessary for this paper. In the standard Western well-tempered chromatic scale, each octave consists of twelve notes, and, for example, transposing up a minor third means shifting every note three places in the scale. In this context, one typically treats the notes being defined using modular arithmetic, since that moving twelve halftones higher returns one to the same note, albeit shifted by an octave.

In this paper, we introduce some methods for using instead multiplicative groups associated with modular arithmetic as a tool for music composition. There are two broad categories that we discuss here. The primary one is the use of such groups to transform musical themes as tool for composition (development of individual lines as well as for creating harmonies). The second is the role such groups can play in creating new scales from larger collections of notes.

The Groups We Need

Two types of groups associated with modular arithmetic appear in mathematics, one additive and the other multiplicative. The additive one is more widely known. Given an integer r, the value of $r \bmod N$ is equal to the remainder when dividing r by N; thus, for example, $7 \bmod 5 = 2$. Integers are equal to each other mod N if they differ by an integer multiple of N. If we consider the collection of numbers from 0 to $N-1$, these numbers form a group under addition mod N. The number 0 forms the group identity, while for $a \neq 0$, the numbers a and $N-a$ are inverses of each other. These groups are known as the *cyclic groups* [2].

There is another set of groups that use modular arithmetic which are based on multiplication [1]. Given a positive integer N, we can form a multiplicative group Γ_N, whose elements are the integers between 0 and N relatively prime to N (i.e., whose greatest common divisor with N is 1), and with the binary operation of multiplication mod N. If N is prime, Γ_N contains $N-1$ elements; for example, Γ_5 has the elements 1, 2, 3, and 4, with multiplication rules like $2 \times 3 = 1$ and $3 \times 3 = 4$, since all arithmetic is done mod 5. For composite N, Γ_N contains fewer than $N-1$ elements. For example, the elements of Γ_{12} are 1, 5, 7, and 11, with multiplication rules such as $5 \times 5 = 1$ and $11 \times 7 = 5$. The cyclic groups, based on modular addition, have seen regular use in Western music. Here, we introduce uses of the multiplicative groups for modular arithmetic, which provide a very different set of tools.

Transforming Musical Lines

In Western classical music, there are certain standard techniques that rely on treating notes as elements of an additive group. Transposition is one of these; one takes a set of notes and then shifts them up or down a precise number of half steps. This is used in the development of themes, as well as in generating musical lines to be played against each other, such as in a fugue or other type of canon [4]. Another technique resting on the additive point of view (swapping addition with subtraction, i.e., employing additive inverses) is inversion: each half step up is replaced by a half step down (potentially shifted an octave at times) [4].

We offer here a novel way to transform a musical line. Start with the 12-note well-tempered chromatic scale, and assign the notes integer values from 1 to 12, which for now we do in order of increasing half-steps from *C* to *B*. There are twelve notes; observe that twelve is the order of the group Γ_{13}. Consequently, we can pair each of the numbers from 1 through 12 with its inverse under Γ_{13}, and then pair the corresponding notes. The result pairings are shown in Figure 1. Note that these pairings are symmetric (e.g., *D* maps to *A*♭, and *A*♭ maps to *D*), and incorporate a mix of intervals: unison/octave, minor third/major sixth, major fourth/major fifth, and tritone.

C	*C*♯	*D*	*E*♭	*E*	*F*	*F*♯	*G*	*Ab*	*A*	*B*♭	*B*
C	*F*♯	*Ab*	*A*	*G*	*B*♭	*C*♯	*E*	*D*	*E*♭	*F*	*B*

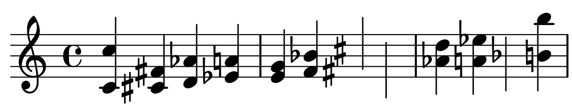

Figure 1: *Chromatic scale, first transform: notes paired vertically; shown in tabular and musical form.*

There was nothing special about starting our chromatic scale on *C*; we could label the notes from 1 to 12 starting with any note in the chromatic scale. Each of these produces a distinct mapping of the original scale. We call these transformations the Γ_{13} multiplicative duals; the full list is presented in Figure 2.

	C	C#	D	Eb	E	F	F#	G	Ab	A	Bb	B
Transform 1	C	F#	Ab	A	G	Bb	C#	E	D	Eb	F	B
Transform 2	C	C#	G	A	Bb	Ab	B	D	F	Eb	E	F#
Transform 3	G	C#	D	Ab	Bb	B	A	C	Eb	F#	E	F
Transform 4	F#	Ab	D	Eb	A	B	C	Bb	C#	E	G	F
Transform 5	F#	G	A	Eb	E	Bb	C	C#	B	D	F	Ab
Transform 6	A	G	Ab	Bb	E	F	B	C#	D	C	Eb	F#
Transform 7	G	Bb	Ab	A	B	F	F#	C	D	Eb	C#	E
Transform 8	F	Ab	B	A	Bb	C	F#	G	C#	Eb	E	D
Transform 9	Eb	F#	A	C	Bb	B	C#	G	Ab	D	E	F
Transform 10	F#	E	G	Bb	C#	B	C	D	Ab	A	Eb	F
Transform 11	F#	G	F	Ab	B	D	C	C#	Eb	A	Bb	E
Transform 12	F	G	Ab	F#	A	C	Eb	C#	D	E	Bb	B

Figure 2: *All twelve chromatic* Γ_{13} *multiplicative dual transformations. Notes are paired vertically.*

This now forms a novel tool for transforming musical themes. I have experimented with using the Γ_{13} multiplicative duality to transform familiar melodies and to create duets in which the two lines are Γ_{13} multiplicative duals of each other. While the space of this paper does not allow detailed elaboration, the resulting music manages to be highly achromatic while still not seeming entirely unfamiliar. As samples, I present, in Figure 3, the first few measures of the melody of Beethoven's *Ode to Joy*, along with three of the twelve possible Γ_{13} transforms of this phrase. It would be interesting to see what these transformations might yield in the hands of an experienced composer; I am particularly interested in the possibility of canons in which the additional voices that enter are selected from the Γ_{13} multiplicative duals of the original voice.

One interesting feature of these transformations is that each preserves one major chord (leaving the root unchanged, and swapping the major third and perfect fifth) and one minor chord (swapping the root and the

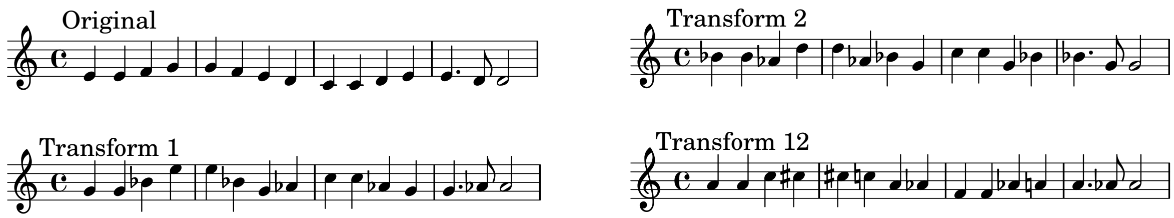

Figure 3: *The initial measures of* Ode to Joy*, and three* Γ_{13} *transforms thereof.*

minor third, and leaving the perfect fifth unchanged, so transforming the root form into the first inversion); for example, in the first dual worked out, the C major and E minor chords are preserved. Exploring the transforms of other chords reveals additional properties. For instance, again using our first dual as an examplar, we see that this transformation maps the notes of the A minor and C minor chords into each other, while the A major chord goes to the dissonant combination E♭/F♯/G. By judicious choice of which Γ_{13} multiplicative dual transform to apply to a piece in a conventional major or minor key, one can either preserve or mask original harmonic structures and create more or less dissonant (by conventional standards) results.

The Γ_{13} dual transforms also offer a way to generalize Schönberg's twelve tone methodology [6]. In 12-tone music, one first writes a sequence of all twelve notes of the chromatic scale, and then performs transformations on this initial tone row—in its simplest form, using (combinations of) inversion, retrograde, and transposition transformations—to create music which is not in a particular key, and in which all notes receive equal weighting. We now have a new option: transform the original tone row via any of the twelve Γ_{13} multiplicative dualities. The nature of these transformations allows the resulting generalized twelve-tone pieces to be even more liberated from tonal experiences of music. (Of course, one could also combine any of the Γ_{13} transformations with the standard transformations of twelve-tone music.)

We close by noting that all the above methodologies can be applied to other scale systems, provided that the number of notes in the scale is one less than a prime number. For example, the whole tone scale (e.g., C, D, E, F♯, G♯, A♯) has six notes, and so the group Γ_7 can be used in conjunction with this scale.

Subscales

In the group Γ_N, the elements are those numbers from 1 to $N-1$ that are relatively prime to N. If N is prime, then all the numbers from 1 to $N-1$ are included, but if N is composite, this is not the case. For composite N, what might be an application of Γ_N to music?

To lay the groundwork for our answer, consider the relationship between the twelve notes of the chromatic scale, and the seven notes of the D major scale. The D major scale can be viewed as a subset of the chromatic scale, or, if you like, a *subscale*. To be sure, the D major scale can be treated in its own right, but there is a certain logic to seeing it as a subscale. If we transpose the D major scale so that it starts on another note that is within the D major scale, that new scale will have some notes that were not in the original D major scale. However, at least when we use well-tempered scales, all such transpositions will always be covered by the notes of the chromatic scale.

Here, we will look at new subscales that can be created from the various modes. Consider first the mixolydian mode, represented here by the notes G, A, B, C, D, E, F, and then G again at the octave. (For our purposes here, it useful to include the foundational note at the start and the end, just as we would play the scale.) There are eight notes here, and although 9 is not a prime number, we can ask if there is a meaningful way to apply Γ_9, the multiplicative group modulo 9, to this scale.

Upon trying, the answer is immediately apparent. Let us assign numbers to these notes sequentially, with 1 corresponding to G, 2 to A, and so forth, leading to 7 for F and 8 for G. If we try to apply Γ_9 to this scale, we must recognize that not all integers from 1 to 8 are included; the elements of Γ_9 are 1, 2, 4, 5, 7, and 8. The corresponding notes are G, A, C, D, F, and then G again. Restricting to these notes, we get the familiar pentatonic scale, the one that corresponds to the same intervals as are found among the black keys of a piano; this particular ordering is known as the Egyptian scale [3].

So one way to understand this pentatonic scale is as the subscale that emerges from the application of Γ_9 to the chromatic scale. We could use the mutiplicative inverses within Γ_9 to manipulate music written in this pentatonic scale just as we used Γ_{13} in the preceding section. One might worry that the note G corresponds to two different numbers, 1 and 8, but that poses no problem here, as within Γ_9, both 1 and 8 are their own inverses, and so the dual of G is unambiguously still G.

Of course, the mixolydian mode was just one example. For each of the standard modes, we can perform this Γ_9 projection to get an associated pentatonic scale. Among the eight modes, three lead to the Egyptian scale, but the other modes produce different pentatonic scales. In the table below, we show the variety of pentatonic scales emerging via our method when applied to the various modes. The specific notes listed arise from writing each mode in terms of the notes A, B, C, D, E, F, and G (so the Ionian mode starts on C, Dorian on D, etc.), with the names [3] of the resulting pentatonic scales as appropriate.

Table 1: *From Mode to Pentatonic Scale*

Original Mode	**Resulting pentatonic scale**	**Name of pentatonic scale**
Ionian	C, D, F, G, B, C	[name undetermined]
Dorian	D, E, G, A, C, D	Egyptian scale
Phrygian	E, F, A, B, D, E	notes of Japenese In Sen scale
Lydian	F, G, B, C, E, F	notes of Balinese scale
Mixolydian	G, A, C, D, F, G	Egyptian scale
Aeolian	A, B, D, E, G, A	Egyptian scale
Locrian	B, C, E, F, A , B	notes of the Japanese Hirajoshi scale and of the Chinese scale

Summary and Conclusions

We have presented here methods for employing multiplicative group theory in music in contexts in which additive transformations have conventionally been used. This opens up new techniques for musical composition. Furthermore, because the methodologies can easily be generalized to scales of different numbers of notes, there is a rich array of possibilities to be explored. One particularly intriguing feature is that while transformations like transposition and inversion have a very similar form regardless of whether one's scale has 12 or 28 notes, the methods discussed here based on Γ_N multiplicative duality have very different forms if one works with Γ_{13} as opposed to Γ_{29}. We are eager to see how these transformations manifest in non-conventional scales, whether microtonal or non-octave.

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