Visualizing Symmetry Subgroup Structures Using Simple Motifs

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Abstract

Symmetric patterns can be understood mathematically as the resulting action of a symmetry group on a base motif. In each symmetry group, all its elements can be represented by transformation matrices. Using the subgroup structure of a base symmetry group, patterns can be created that have some integration into the overall symmetry. Examples of this process are shown for two dihedral groups and a wallpaper group.

Introduction

Throughout history, symmetry has been used to create interesting patterns and objects. Many objects contain patterns with symmetry groups of various types. For example, automobile wheels might have 5-fold, 6-fold, 7-fold, or higher symmetry in their main body. However, many have 5 lugnuts symmetrically placed at the center. Thus the symmetry structure will match when the wheel symmetry is also a multiple of 5. Another application area is in decorative dinnerware. Plates and bowls can be decorated with rings of different symmetry types [2]. Similarly, decisions on the use of patterns fabric often depends on how the pattern relates to the object being created (such as clothing and furniture). An artist often needs to choose which symmetries to feature and which to minimize.

Mathematically, these patterns can be understood as the resulting actions by the elements of a symmetry group on a base motif. A group is a set G together with an operation $\star$ such with several special properties. First, there exists an identity element e in G such that $a \star e = e \star a = a$ for all elements a in G. Second, for every element a in G, there exsits an inverse element b in G such that $a \star b = b \star a = e$, where the element b is often denoted a^{-1}. Third, the operation $\star$ is associative, so that $a \star (b \star c) = (a \star b) \star c$ for all elements a, b, and c in G. Finally, the set G is closed under the operation $\star$, so that $a \star b$ is in G for all elements a and b in G.

For a symmetry group, the elements can all be represented by 4×4 augmented (or affine) transformation matrices as typically used in computer graphics [4]. For the patterns on the plane, the z component is simply equal to zero. The group is a set of matrices defining each of the symmetry operations; the set may be infinite, but can be restricted to a finite region in practice. The operation $\star$ is matrix multiplication, which represents ordered compositions of symmetry operations. Given a motif, the matrix elements of the group are applied on the motif and the result of that action is noted. Applying every matrix element of the group (every symmetry transformation) results in the full symmetric pattern.

However, it is often of interest to restrict the symmetries within the context of a larger symmetry group. For example, a chess or checkerboard pattern with alternating colors results from restricting the symmetries found in a quadrille pattern. A subgroup is a subset of the symmetry group that is a restriction that uses only a portion of the symmetry group elements while maintaining closure under the base group. The number and type of subgroups is related to the base symmetry group. More formally, a subgroup H of a group G is a subset of G such that is a group with respect to the opertion $\star$. Two trivial examples of subgroups include the subset containing just the identity element and the entire group G.

In the case of symmetry patterns fixing a point, the group structure is either kaleidoscopic and represented by a dihedral group D_n or gyroscopic and represented by a cyclic group C_n, where n represents the smallest rotational unit of $360°/n$ found in the pattern. These symmetry groups are associated with the regular n-sided polygon. The group C_n consists of rotations about the origin by integer multiples of $360°/n$. The group D_n consists of all the elements of C_n plus n equally spaced reflections through the origin. Thus, the group C_n is a subgroup of D_n; there are typically many more subgroups, depending on the divisors of n. Given any group, the number of elements of any its subgroups must evenly divide n, by Lagrange's theorem [3].

Methods and Results

The power of mathematical groups is that they can be applied to many situations, not just the understanding of symmetry. However, algebraic manipulation alone does not always translate into intuition about groups. The goal of this work is to describe a framework for the visualizing and gaining intuition about the symmetries possible in an overall design given a base symmetry group. Using the subgroup structure of the base symmetry group will result in all subgroups having some integration into the overall symmetry.

Consider the group D_4, the group associated with the symmetries of the square. It contains 8 elements, thus the only possible subgroups have 1, 2, 4, or 8 elements. There is exactly one cyclic subgroup for each of the orders 1 (the identity), 2 (identity and 180° rotations), and 4 (identity and rotations by 90°, 180°, and 270°). Additionally, there is one dihedral subgroup of order 8 (the original group D_4), two dihedral subgroups of order 4 (the first containing both horizontal and vertical reflections; the second containing 45° diagonal and −45° diagonal reflections), and 4 dihedral subgroups of order 2 (each containing one of the reflections in the D_2 groups plus the identity).

One can use a family of motifs to help visualize the relationships among the symmetry subgroups of D_4. If the motifs have a cohesive theme with a different motif for each subgroup in the lattice structure, then a single resulting image will result that contains features of all possible symmetries. An example of faces where facial features are repeated using subgroups of the base D_4 symmetry is shown in Figure 1. The features and symmetries are as follows: eyes and ears, D_4; heads, C_4; eyebrows, D_{2a}; noses, D_{2b}; hair, C_2; mustache, D_{1a}; lower earings, D_{1b}; mouths, D_{1c}; moles, D_{1d}; and neck, C_1. The four axes associated with the reflection elements of the group are shown as dotted lines.

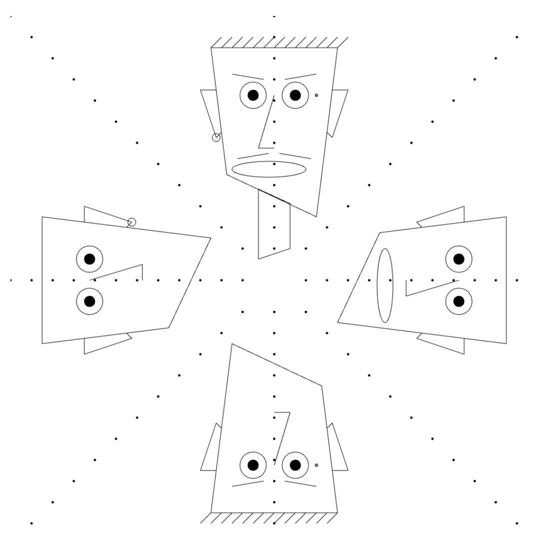

Figure 1: *Facial features drawn according to the subgroups of D_4. See text for description.*

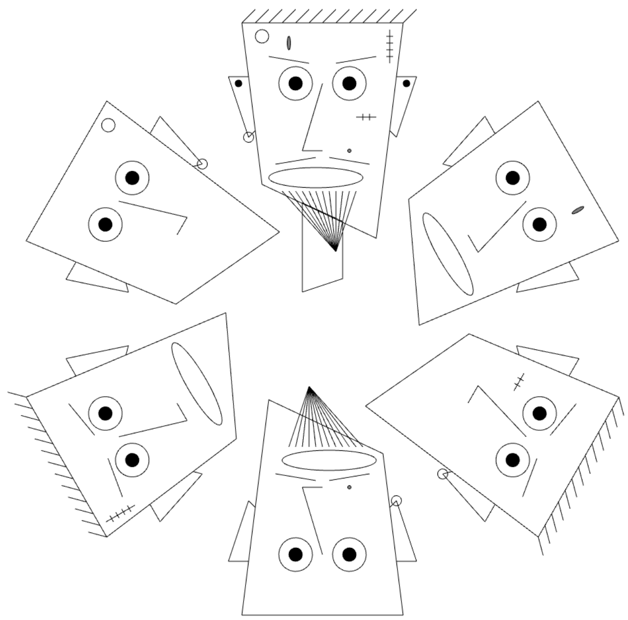

Figure 2: *Facial features drawn according to the subgroups of D_6. See text for description.*

Similarly, one can visualize the subgroups of D_6 as shown in Figure 2. The features and symmetries are as follows: eyes and ears, D_6; heads, C_6; eyebrows, D_{3a}; noses, D_{3b}; hair, C_3; mustaches, D_{2a}; lower earings, D_{2b}; mouths, D_{2c}; beard, C_2; moles/tattoos/scars, D_{1a}–D_{1f}; and neck, C_1.

In addition to cyclic and dihedral groups, the same concepts can be applied to wallpaper group symmetries. These groups cover the plane with an infinite number of horizontal and vertical translations, thus they are infinite and will have an infinite number of subgroups. One can still gain some insight by restricting these groups to a finite region of the plane. Associating the left and right edges as well as the top and bottom edges using modular wrapping in the horizontal and vertical directions results in a finite group with a finite number of subgroups. This related finite group will have similar visual characteristics to the infinite group. An example of the wallpaper group 442 (p4) is shown in Figure 3. The resulting group contains 128 elements and 256 subgroups. The petals and center of the flowers were generated using 5 of the subgroups; the identification of the subgroups used is left as an exercise for the reader.

Discussion

While not explicitly shown in this paper, the same concepts can be applied to frieze and soccer-ball symmetry groups. In addition to being a design tool, this concept can be used to help students explore groups and subgroup structures, providing insight and intuition, especially with more complex concepts such as normal subgroups and stabilizers. A playful technique would be to construct cards or a game board where one creates certain types of symmetry.

For a general dihedral group, Cavior's theorem [1] states that for $n \geq 1$ the number of subgroups of D_n is $\tau(n) + \sigma(n)$, where $\tau(n)$ is the number of positive divisors of n, and $\sigma(n)$ is the sum of positive divisors of n. This is because the subgroups of D_n are either cyclic or dihedral and the number of cyclic subgroups of D_n is $\tau(n)$ and the number of dihedral subgroups of D_n is $\sigma(n)$. For example, the divisors of 6 are 1, 2, 3, and 6, so $\tau(6) = 4$ and $\sigma(6) = 12$. The complexity of the subgroup structure of D_n generally increases with n,

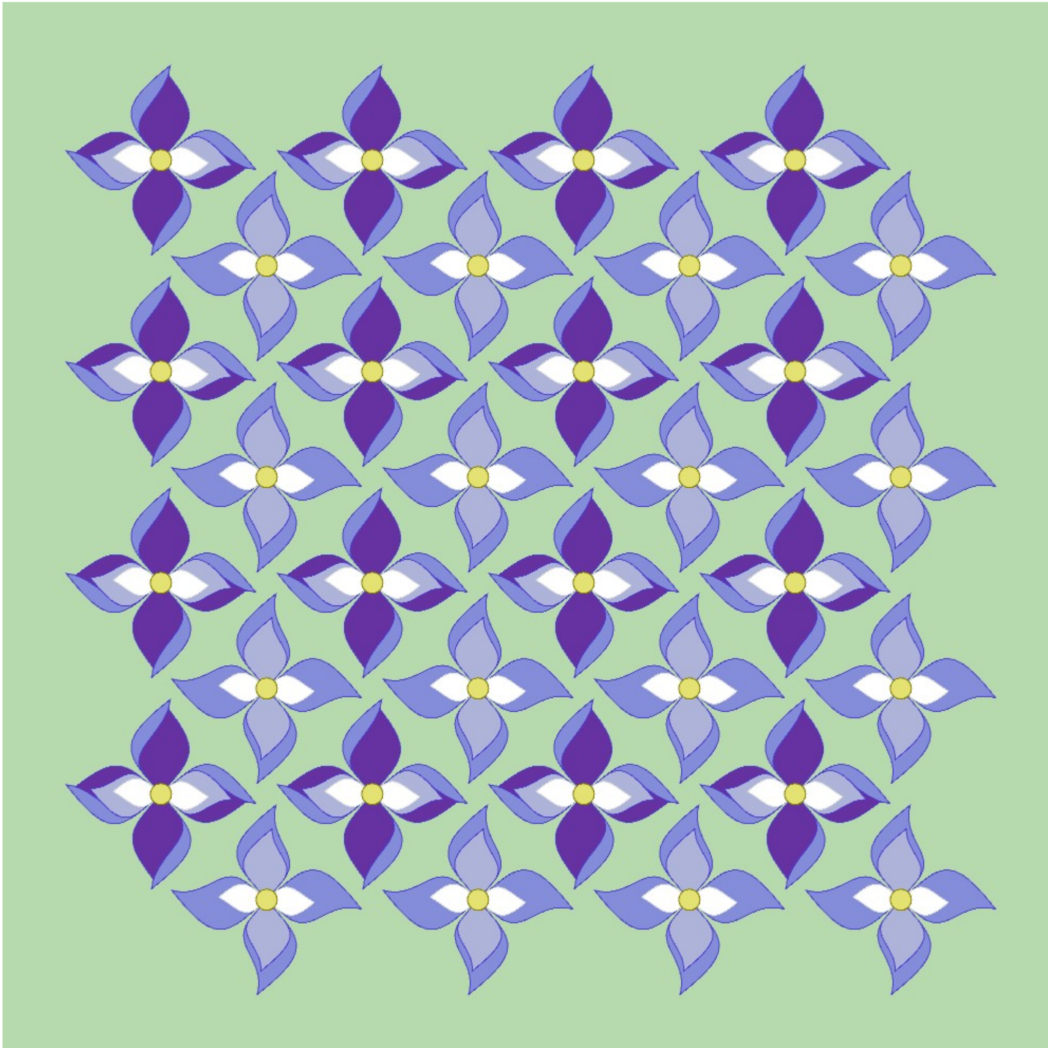

Figure 3: *Flowers using some of the normal subgroups of a finite version of the symmetry group 442 (p4). See text for description.*

which may require an impractically large number features to fully visualize the subgroup structure. Wallpaper and frieze patterns with large repeats will also result in groups with very large numbers of subgroups, thus potentially limiting the groups which can be easily explored.

References

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