Catenary Arch Constructions

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Abstract

The catenary is a beautiful curve, with important applications in science, engineering, and architecture. In our hands-on workshops, participants build catenary arches while discovering something about their mathematical properties and the design choices that led to this construction. Three materials are explored: paper, cardboard, and wood. The large wooden arch is designed to be displayed in public spaces and to make viewers aware of the link between art and math. As part of our Making Math Visible project, free templates and three-part lesson plans are available online.

Introduction

In our ongoing pursuit to find beautiful ways to make math visible, [1] we were inspired during a visit to St. Louis, to create our own artistic rendition of the engaging catenary form. Eero Saarinen's majestic arch seemed like a natural gateway for inviting viewers to think about math, to ask questions, and to ignite mathematical discussion. Figure 1 shows the result of our project, a wooden walk-through catenary arch that anyone can replicate by using the instructions and laser-cutting templates we have created. [2]

Figure 1: *Catenary arch of laser-cut wood (stained) and cable ties, 6.5 feet tall.*

The St. Louis Gateway Arch is a giant mathematical drawing in the sky. Every day, thousands of visitors are attracted by its beauty, compelled to step inside and to physically follow its arc upwards. The inviting form of the catenary has fascinated humans for centuries, going back to Galileo who famously puzzled over its mathematical properties. After the invention of calculus, both the upward orientation of a catenary arch and the hanging form of a chain could be precisely understood. The suspended version of a catenary can be found all around us, from the delicate threads of a spider web, to the ubiquity of telephone wires, to the rusted anchor chain of a battleship. In contrast, true catenary *arches* are relatively rare; before Eero Saarinen's monumental St. Louis example, they mainly occurred when stone masons or architects such as the visionary Antonio Gaudi, consciously upended the familiar hanging shape. [3, 4, 5]

We conceived of various ways that we could add an artistic body to the underlying mathematical bones of the curve. The pure catenary is just a line of zero thickness, but we needed to flesh out its substance so it could stand up and have the presence and physicality of a sculptural object. The first step was to design software that allowed us to manipulate many parameters and make artistic decisions as we viewed various hypothetical renditions on the screen. This led us to produce templates for a paper version that can be assembled as a real concrete object. From there, it was a natural step to scale it up to a larger cardboard arch, which convinced us of the worth of our artistic choices. We then created a full size wooden sculpture large enough to walk through. It stands on its own as an artistic manifestation of the mathematical concept.

One aspect which makes the catenary arch so captivating is that it has deep mathematical as well as artistic beauty. It can be enjoyed for its elegant clean lines and aesthetic minimalism or for its fascinating role in mathematical physics. A hands-on model can serve as a wonderful educational tool to demonstrate that the catenary is the optimal shape for an arch that carries its own weight. It has been used as an exhibit in many science museums, where visitors can build an arch with self-supporting blocks. When constructing it, the builder discovers that the weight of the arch is directed along its length, so no sideways force exists to slide the blocks apart where they rest on each other. This requires a beautiful curve, similar to a parabola, yet subtly different. Educators will find this to be a natural opportunity to explain that a parabola is a quadratic polynomial, while the catenary is in the exponential curve family. Fully understanding why the catenary is the shape of a hanging chain and has the optimal properties it does requires drawing a force diagram then applying a bit of calculus, something that is generally omitted in science museum exhibits. A complete derivation can be found in university-level math and engineering textbooks or conveniently online at Wikipedia. [5]

The educational possibilities presented by the catenary arch naturally led us to incorporate it into our Making Math Visible project. Its mathematical depth lends itself well to hands-on activities that illustrate the intersection of math and art. Our templates and lesson plans are freely available online [2] and we welcome people to recreate them in classrooms, schools, and public spaces. Educators will find that they have many entry points, which allows them to adapt the activities to the level of the participants. We use a three-part lesson model to first activate students' thinking in a "minds-on" exercise, followed by the main construction activity, and concluding with a consolidation to ensure mathematical concepts are clearly understood and retained.

Design and Customization

Customizing the catenary is a good exercise in 3D design iteration. We wrote a small program in Mathematica [6] to render many variations for the skin of the arch, allowing us to explore a sizable design space while thinking about aesthetics, strength, complexity, size, ease of assembly, and the constraint that the largest piece must not be too large to fit on our laser-cutter. In the end, after all the parameters were tweaked to our liking, the program output the face templates in a format we could adapt for laser-cutting.

Figure 2a shows a rendering of a possible design of a thirteen-section arch in which the tube has a constant square cross section. It is surprisingly ugly and blocky, in part because it does not taper down to a thinner cross section at the top. Saarinen's design, Figure 3, is elegantly tapered, making it feel solidly anchored to the earth while gracefully reaching for the sky. Figure 2b shows the improvement afforded by a taper, while experimenting briefly with a pentagonal cross section instead of a square. In addition, diamond-shaped openings are introduced.

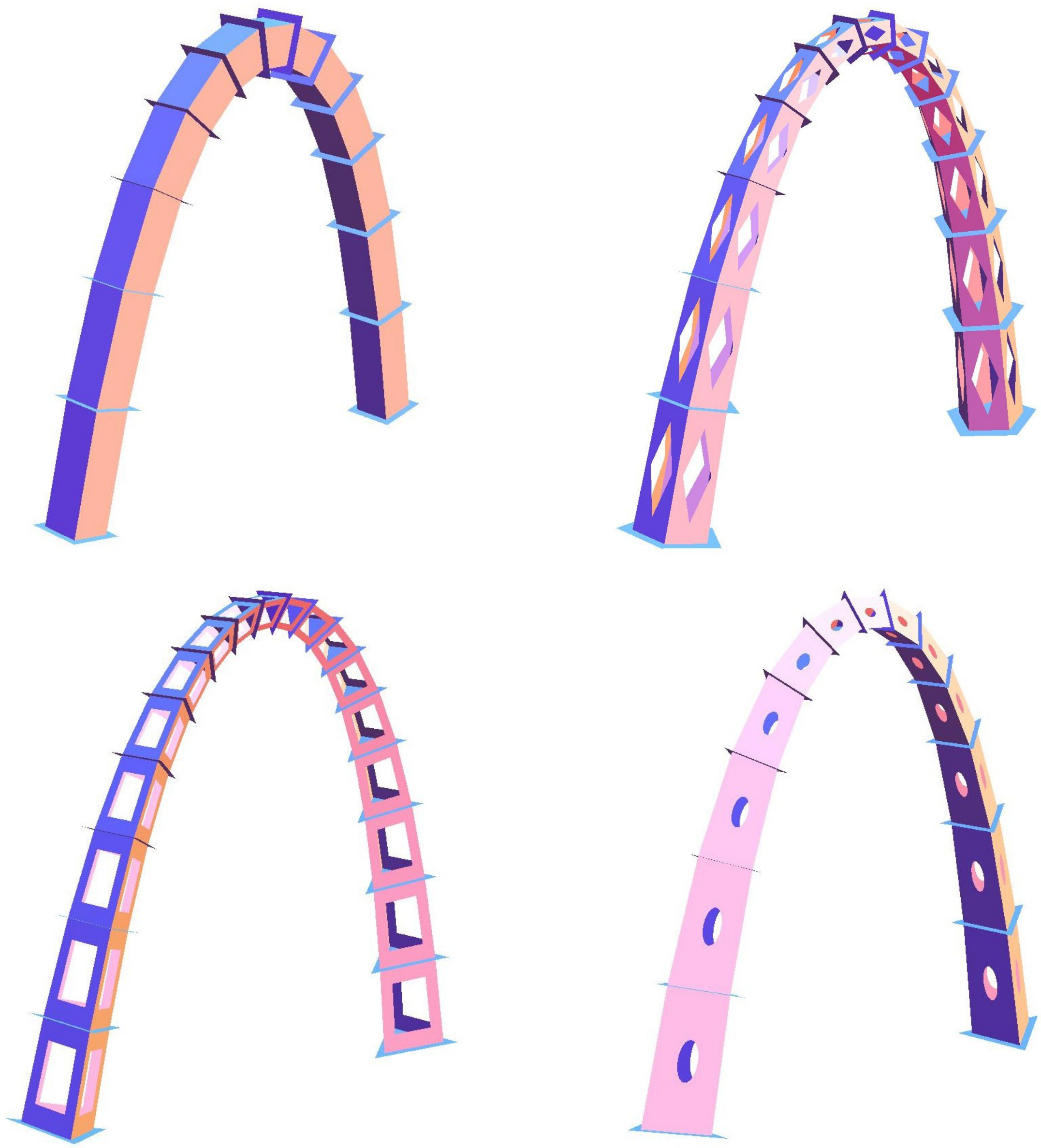

Figure 2: *Four possible realizations of a catenary arch: (a) square in cross-section, un-tapered, with no openings; (b) pentagonal with diamond openings; (c) triangular with rectangular openings and 18 segments; (d) triangular, pointing upward, with small elliptical openings.*

Some sort of openings are required when assembling the wood panels with cable ties, in order to access the interior and to thread the ties through small holes. However, the angularity of these diamond-shaped holes is an unfortunate choice that conflicts with the smoothness of the curve. In Figure 2c, the pentagonal tube idea has been abandoned and simplified to an equilateral triangle, the diamond openings are tamed a bit in becoming rectangular, and the number of segments has been increased to eighteen. These openings are only a slight improvement over the diamonds and the many segments would require significantly more building time, yet do not add much to the smoothness of the curve. However, Saarinen's choice of a triangular cross section is clearly an aesthetically pleasing simplifying step. In Figure 2d, we experiment with the rotational "phase" of the cross section by briefly considering twisting the triangle so there is a ridge line along the outside and peak of the arch. This feels overly sharp and knife-like, again confirming Saarinen's eye for design. We also experimented with elliptical portholes minimally sufficient for accessing the interior, however their small size was awkward and distracted from the overall design. In the end, we simply had no choice but to copy the tapered, triangular cross section, phased as in St. Louis. We enlarged the elliptical windows, making them light and open to allow interior access, while structurally reinforcing the corners as fillets.

An interesting design question lies in the lengths of the individual segments. As we were working with flat pieces of plywood, the underlying smooth curve had to be approximated with discrete polygons. Even after deciding on the number of sections, the question remained of precisely where the slices should be made. For visual reasons, the sections near the top, where the curvature is greater and the cross section is thinner, had to be shorter, but how much shorter? There are a number of ways one might try to approach this rationally with a mathematical principle of design. Instead we cobbled together an interpolating function that smoothly varied between a short-enough segment at the apex and one that just filled the 12-by-24-inch bed of our laser cutter at the bottom.

Another set of design issues concerns the triangular bracing pieces which serve as "floors" where the thirteen modules join. In our wood catenary, there are fourteen of these, including the two ends, analogous to the fourteen fence-posts required to hold thirteen lengths of fence. (In the paper and wood versions, there are twenty-six triangles, as each end of each module is capped individually.) They are smaller near the top, in proportion to the taper. Functionally, they serve to align the side panels and lock everything rigidly. We believe they are necessary for strength, but haven't experimented with omitting them. Visually, they emphatically demarcate the straight sections, making explicit the discrete approximation to the underlying curve. Each slice is orthogonal to the catenary, except the two extreme cuts, which are horizontal, so the arch can rest stably on the floor. As a practical matter, the wood triangles are hollow, which allows them to be nested for laser-cutting and provides additional access to the interior during the assembly.

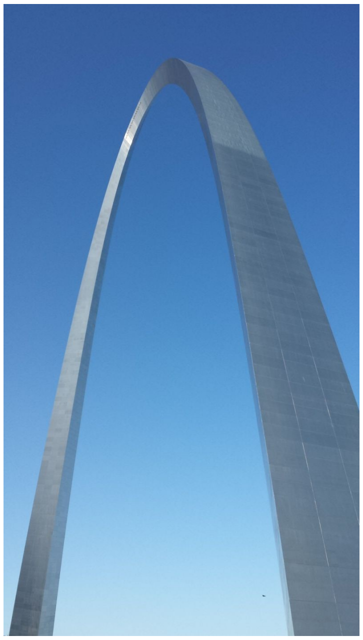

Figure 3: *Gateway Arch in St. Louis by Eero Saarinen, 630 feet tall.*

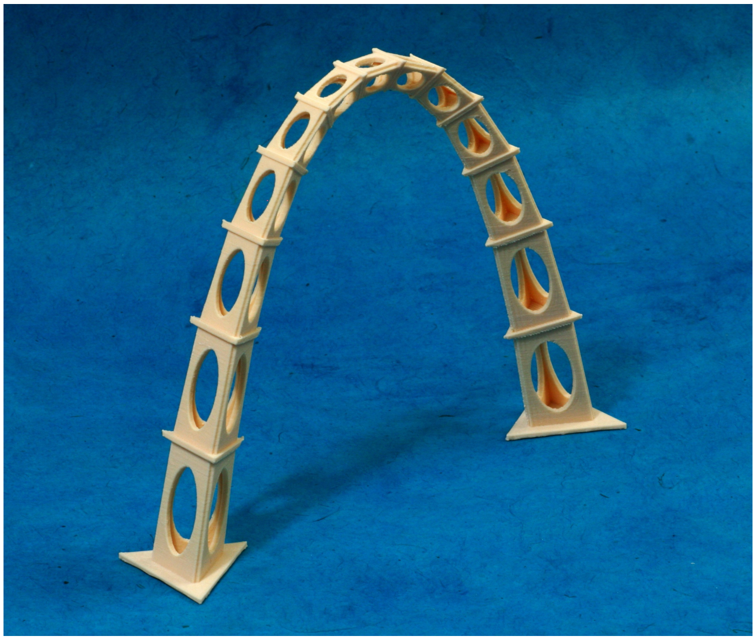

Figure 4: *3D printed model of our final catenary design, 6 inches tall.*

An additional engineering detail is that although the arch balances beautifully on its own, we designed a simple base on which to attach the large wood version. This provides stability and safety if people bump into it. The base is basically just a sheet of plywood with small holes for cable ties to loop through, positioned to replace the two horizontal triangle brace pieces at the bottom of the arch. It also serves to fix the position of the two ends with the proper spacing.

Finally, with the design programmed into Mathematica, it was relatively straightforward to also produce an STL file suitable for 3D printing a miniature model of the design. Figure 4 shows a six-inch tall plastic version made on a Makerbot fused deposition machine. The STL file can be downloaded and replicated as an educational model. [7]

Paper Catenary

Heavy paper such as card stock is an inexpensive medium that is versatile, widely available, and easy to work with. Students of all ages can make surprisingly complex structures just using paper, scissors, glue, and tape. The first of the three versions presented here is a paper arch, 18 inches tall (48 cm), made of thirteen modules. Assembling the pieces and balancing them provides a fun dexterity challenge that can be adapted to different skill levels. Getting it to stand requires very precise fabrication and collaborative construction with four or more hands. Given our access to laser cutters, we were able to fabricate precisely cut pieces with elliptical openings. If working with scissors however, the openings could be omitted. Assembling it is an exercise in teamwork and communication that conveys a sense of the importance of careful engineering.

When we have tried this with student groups, we have observed their enormous satisfaction upon completion. Figure 5 shows the separate pieces under construction and Figure 6 shows the paper catenary in its assembled form. Note that the thirteen separate pieces are just resting on each other, held together by their own weight, not glue or tape. A light touch or breeze will cause the construction to collapse. Participants typically require many attempts before they can get everything to balance just right, which makes the moment of success all the more joyful.

Figure 5: *Paper catenary assembly.*

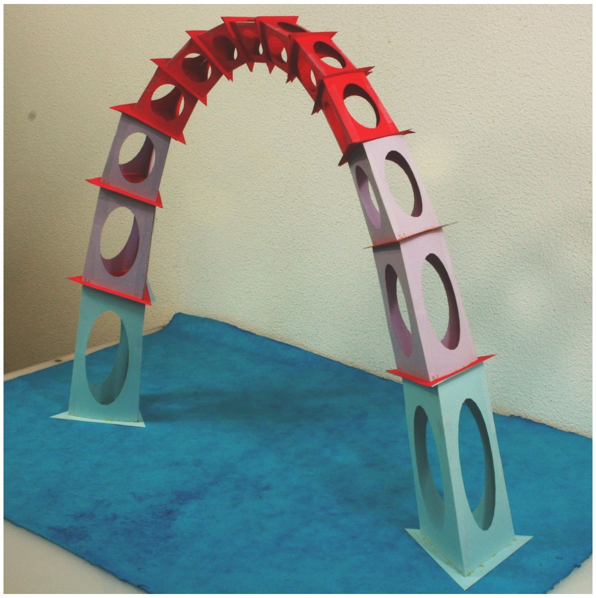

Figure 6: *Paper catenary assembled.*

We developed a detailed three-part lesson plan that teachers can access and adapt to their student's level. It includes a complete materials list and the necessary templates. The workshop can be presented as a simple hands-on construction that merely familiarizes students with the notion of a catenary, or it can be a much deeper foray into mathematics, for example, comparing the formula of a parabola with the formula of a catenary and considering their asymptotic differences.

Cardboard Catenary

In our experience developing hands-on activities, we have repeatedly observed that participants greatly enjoy applying newly acquired knowledge to larger-scale challenges. Building a scaled-up version of a project like the paper catenary consolidates understanding and increases engagement. So it was natural for us to design a human-size arch, which led us to choose cardboard as the appropriate material. Cardboard is a surprisingly strong, durable, and affordable medium. We have found that if students have already made the paper version, you can simply hand them the cardboard pieces with some rolls of packing tape and they need no further instruction. It is a natural example of linear scaling and proportional reasoning. A single four-foot tall cardboard catenary suffices for a class-size group, because the thirteen modules can be built in small teams. The result is a challenge that can be delightfully knocked down and rebuilt over and over again, much like a science museum exhibit. Figure 7 shows that the cardboard catenary can remain as a natural accoutrement in a classroom setting throughout the year.

Accurately cutting cardboard sheets into the required components is best done with a laser cutter. We realize that many educators do not yet have access to this technology, but we are optimistic that laser cutters will become more and more available in the near future. Meanwhile, the template can be traced and cut with a knife.

Figure 7: *The cardboard catenary, 4.5 feet tall, fits well in a classroom environment.*

Wood Catenary

Wood offers more substance and permanence than mere paper or cardboard when making a work of art. It provides strength and a surface texture that adds to the overall visual and tactile appeal. Wood can be beautifully stained, adding yet another dimension to the visual experience. We chose to use laser-cut Baltic birch plywood to construct a seven-foot tall catenary arch that is an homage to Saarinen's sculpture in the sky. Figure 8 gives a sense of its graceful presence in a natural setting. Its fluid line seems to effortlessly emerge from and re-enter the Earth, while the ellipsoidal openings are suggestive of the chain from which the catenary was originally conceived.

While based on the same underlying design as the paper and cardboard arches, this version is permanently held together. We conceive of it more as an artwork to be exhibited in a gallery or sculpture garden than as a puzzle to be repeatedly assembled. The components are easily portable, so it can be erected as a sort of pop-up math/art installation that gets people engaged in mathematical conversations. During construction, its modules are joined with cable ties, but afterward the beauty of the underlying math is intended to be appreciated visually, rather than as an educational hands-on activity. Figure 9 shows that it also makes a great entryway into a reading nook in a classroom.

In Figure 8 the arch is unstained, giving it a very natural look. However, we have found that applying water-based stain, as shown in Figures 1 and 9, adds a certain richness and visual impact.

Figure 8: *Wooden catenary on the beach at sunset. 6.5 feet tall.*

Figure 9: *Wooden catenary as entrance to classroom reading nook. 6.5 feet tall.*

Summary and Conclusions

We look at the world as artists, but always maintain our educational lens. Catenaries and parabolas are usually experienced as line drawings in the world of abstract mathematics. Saarinen's genius lies in part in creating a solid, tangible, imposing monument, yet keeping the quality and simplicity of a line. Our goal in this project was to keep some of that artistry while giving people a very personal and tactile experience of an abstract mathematical concept on a manageable scale. A physical catenary model should expose the catenary's inherent beauty while preserving its minimalist essence.

Part of the power of mathematical art is how it can penetrate people's field of vision in a casual way and implant mathematical ideas that they may otherwise never encounter. If the catenary curve only stayed on paper in textbooks, the general public would never have opportunity to experience it in any way. Artists like Gaudi and Sarrinen have delighted the public with these forms, however one has to travel to particular sites to see their work. By providing catenary-based experiences through smaller sculptural forms, we hope to familiarize people with the idea that the beauty of math is all around them. They will hopefully become sensitized to see how math, art, and engineering combine in the casual curve of a spiderweb or the everyday dangle of a power cord.

Acknowledgments

We thank all the teachers and students who helped us test these activities.

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