A Class of Spherical Penrose-Like Tilings with Connections to Virus Protein Patterns and Modular Sculpture

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Abstract

The focus of this paper is a class of symmetrically patterned spherical polyhedra we call "Twarock-Konevtsova" (T-K) tilings, which were first applied to the study of viruses. We start by defining the more general concept of a "wrapping paper pattern," and T-K tilings (along with one other virus tiling) are outlined as a subset of them. T-K tilings are then considered as "spherical Penrose tilings" and a pair of algorithms for generating them is described. We believe they present good source material for mathematical sculpture, especially modular origami, having intriguing features such as consistent edge lengths and a limited selection of angles in their constituent faces. We conclude by looking at existing examples of T-K tilings in mathematical sculpture and consider different directions they could be taken in.

Introduction

Symmetrically patterned spherical polyhedra, discovered by geometers, have inspired artists across many cultures [4]. This may well be because, in addition to their aesthetic appeal, they have "structural" benefits: spheres are very stable, and objects with patterns on them, being made of a limited set of repeated pieces, are generally "simple" to construct. For example, the origamist who made the object in Figure 9d only needed to make copies of one simple module and then slot them together. If that same origamist were to make a sculpture of equal size and intricacy but without following a pattern, it would have to involve coordination of different things.

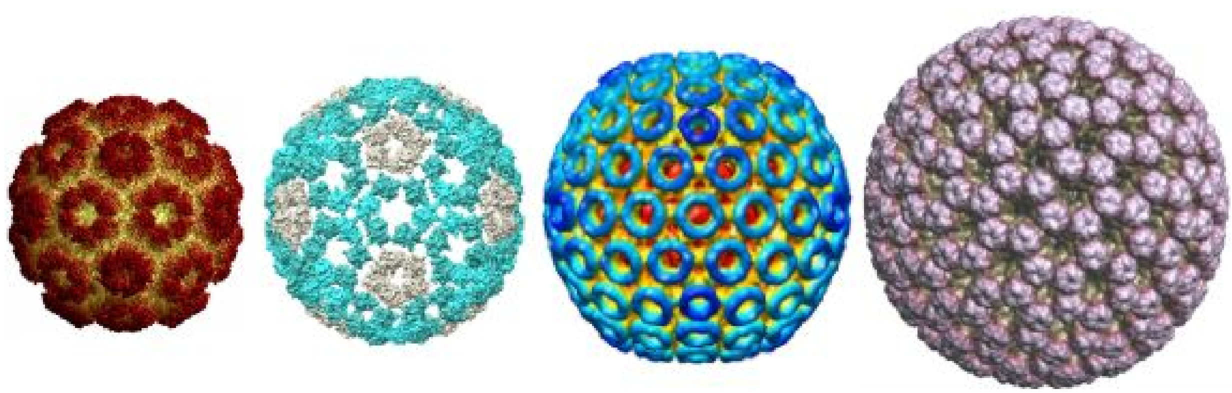

Figure 1: *Four viruses in the "Caspar Klug" [3] group of tilings—proteins are clustered together in either hexagons or pentagons. The patterns are structured in essentially the same way as "Goldberg polyhedra," "buckyballs," "geodesic domes," and there are many mathematical sculptures resembling them [17]. Images courtesy of the EMDB, viperDB and viralzone.*

Patterned spherical polyhedra also happen to appear in the study of virus structure (see Figure 1) and for very analogous reasons. Again, the structural stability offered by spheres has advantages for viruses, and again, being made of a limited set of components is beneficial. Similar to the human "ease of assembly" argument, Viruses have "found" that a *small genome* is more easily transported into host cells. But the smaller an organism's genome is, the fewer different kinds of protein it is able to code for. Therefore, spherical virus shells are usually made of a single-digit-number of different kinds of proteins. To make up the shell, these proteins need to be repeated, creating a pattern (which may also happen to be very aesthetically pleasing) [4]. For more information from a recreational mathematics perspective see [17][16][15].

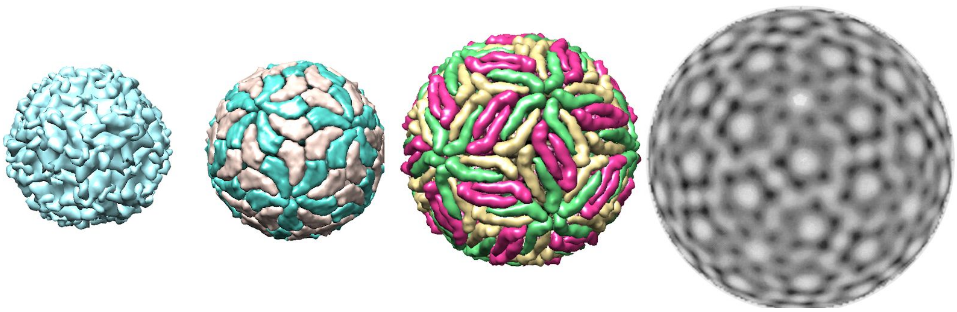

Figure 2: *Viruses that have been proposed to be modelled with Twarock-Konevtsova tilings [9]. The first three are shown in surface representation, while the last —human papillomavirus (HPV)— is shown as a density map. HPV picture from [9] used with permission.*

There is a small number of pattern sets that viruses keep to. We will describe two protein-pattern sets: "Twarock-Konevtsova" (T-K) tilings, Figure 2, and "Caspar Klug" (C-K) tilings, Figure 1. which are the same thing as "Goldberg polyhedra." Other sets exist but have less symmetry, so we do not consider them here. Related groups of viruses will stick to different sets of patterns. Within a certain pattern group a virus can quickly evolve between patterns [1], which is useful because it allows them to change *size* easily. Virus groups, therefore, have a specific mathematical problem to solve: they require a group of spherical patterns that, in some sense, can "scale up and down." C-K and T-K tiling provide a solution to this, and are described in the next section, along with the superset that contains them both, which we call "wrapping paper patterns."

For outreach purposes, we have created an interactive demo that allows the user to control the parameters of C-K and T-K tiling generators. We very strongly suggest that, so that the reader may gain an intuitive understanding of both pattern groups that they examine the generators, which can be seen by going to http://viruspatterns.com [16], and clicking on "Zika virus" when the opportunity arises—or alternatively one can skip the introductory remarks by going to the end of the video, clicking hiv, and then going to the end of the second video. Its source code contains javascript implementation of the algorithms described below.

Definition of "Wrapping Paper Patterns"

Consider a discrete lattice L in $\mathbf{E}^n$. Consider also an n-dimensional polytope homeomorphic to $\mathbb{S}^n$. Let G be the symmetry group of this polytope, and let R be a fundamental domain of G. "Cut out" a copy of R from the polytope and lay it "flat" somewhere on L, with some scale and orientation to be decided. We make a number of copies of R equal to the order of G, and apply G's symmetry operations until we have covered $\mathbb{S}^n$; call the result X.

X is a wrapping paper pattern—a pattern on $\mathbb{S}^n$ which has the symmetries of G and, provided R is not a point, has the structure of L in local regions around each point. We may also want to project X onto $\mathbb{S}^n$, as happens sometimes in this paper. For the rest of this paper let n = 2, as $\mathbb{S}^2$ has the benefits for sculptors described above. Wrapping a Christmas present in a cubic box with periodically-patterned paper is an example of the creation of such an object, hence the name "wrapping paper patterns." The use of patterns akin to wrapping paper patterns to make sculptures has been formally used and described previously in [14], with some of the resulting structures being used as analogies for virus structure [15].

Wrapping paper patterns have been found to be useful for viruses, and for sculptors aiming to make spheres, for much the same reason. Consider taking a set of objects, making copies of them, and for every point in X, placing an object at its location that tries to fit with its neighbours; the objects could be proteins or origami modules. With wrapping paper patterns, the local neighbourhoods of many, possibly all copies, will be very similar, so the process can be "easier" than if the points were randomly arranged.

To get patterns of use to the virus and sculptor, the puzzle is to choose G and R such that X preserves some properties of L. For example, it might be the case that in L, the distance from each point to its nearest neighbour is always some constant value, and we might be able to find some R such that this also holds true of the points on X. The best choice of G and R would be as follows: consider a tiling T on the lattice, made of a finite set of tiles; we would like some X, coming from T, made with G and R, that preserve T's "tile selection." A randomly selected region of such a tiling, with G applied to it, will probably give an output with tiles that do not line up at the edges of the fundamental domain's copies. But if R is chosen carefully, then they might line up better and completely tile the sphere with tiles from T.

To construct a sphere with many copies of a small set of tiles is the ideal situation for the virus, as it requires the virus DNA to code for fewer proteins. And for the sculptor it allows something intricate to be built from simple, copied units, with regularities in 3D that are consistent yet complex.

Caspar-Klug tiling, Figure 3, provides a fairly simple example of a set of wrapping paper patterns that preserve shape-selection. There, G is the subgroup of rotations within the icosahedral symmetry group, which has points of threefold, twofold, and *fivefol*d symmetry, and L is the tiling of the plane by regular hexagons, which has points of threefold, twofold, and *sixfold* symmetry. A C-K tiling needs to be made entirely of pentagons and hexagons. To achieve this, R must be positioned so that its twofold points lie on twofold points of L, its threefold point lies on some threefold point of L, and its fivefold point on a sixfold point of L. We will see that positioning R within T-K tilings to preserve properties of the lattice it comes from is much more difficult, and will be the task of algorithm II.

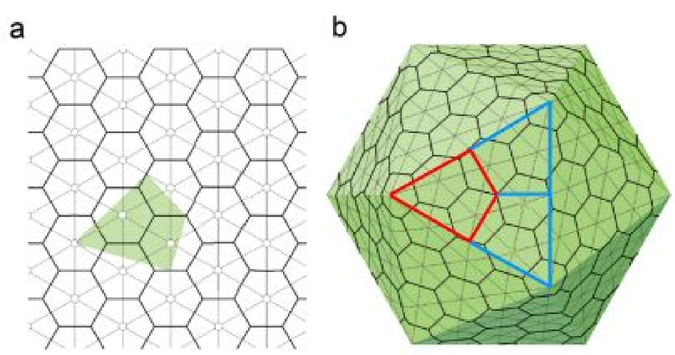

Figure 3: *From [9], used with permission: the construction of C-K patterns, which the viruses in Figure 1 follow;* ***a****: fundamental domain of the chiral icosahedral symmetry group imposed on the hexagon tiling;* ***b****: the result of copying the fundamental domain and applying the group's symmetries.*

Twarock-Konevtsova Virus Protein Patterns Considered as "Spherical Penrose Tilings"

For convenience in the following let $\tau = 2\pi$ and $\varphi = (1+\sqrt{5})/2$.

The question of whether a "spherical Penrose tiling" is a logically consistent idea has been contested, for example on the xkcd forums [6]. User "antonfire" dismisses the idea:

"The Penrose tiling is interesting because it is non-periodic but still, somehow, ordered. For instance, every finite chunk of it reappears in infinitely many places. It's also related by a substitution to a scaled-up version of itself. Neither of these things make sense when you talk about tilings on a sphere."

This is in response to the original poster, an artist, asking whether someone could present a Penrose-tiled sphere. One person suggests using the Riemann sphere to do this, but this is not clearly different to the planar pattern in any satisfying aesthetic way. No progress is made after that, in part due to a refusal to recognize the artist's request as being connected to properly-understood mathematical concepts, because it is not outlined in a rigorous mathematical way.

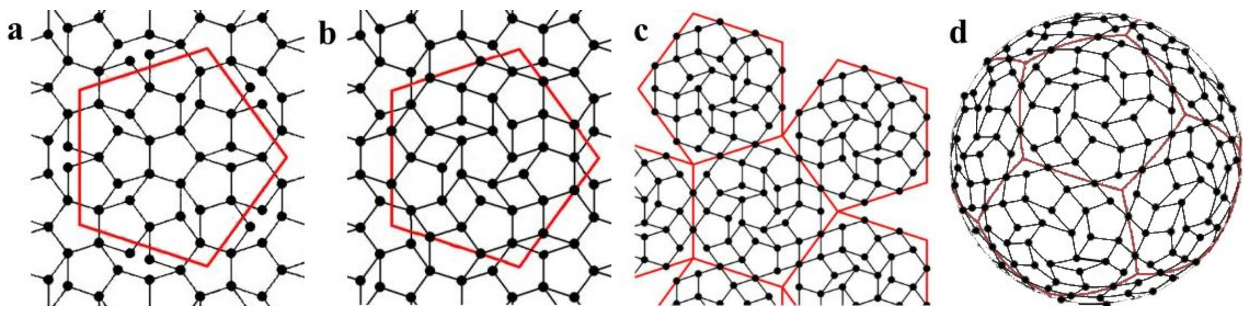

Figure 4: *From [9], used with permission, illustration of the wrapping paper procedure used to obtain a model of papillomavirus, with superimposed pentagons for continuity;* ***a****: a Penrose tiling* ***b****: a modification of the Penrose tiling that serves as the tiling from which the T-K tilings shall be cut (see [9] for details);* ***c****,* ***d****: illustration of how the tiling is imposed on the surface of a dodecahedron. The final result has the same structure as the image of papillomavirus in Figure 2.*

We present T-K tilings as an answer to this question. These patterns were first used to model LA virus and polyomavirus by Twarock [18]; later work by Konevtsova et al used a more consistent formalism and linked them to many more viruses [9]. Mathematically, a T-K tiling is a wrapping paper pattern with a Penrose tiling [11] as the lattice L, and the chiral icosahedral symmetry group as G (though it is easier to think of the dodecahedron instead of the icosahedron). Additionally R is placed on L (Figure 4a) such that the edge lengths of the shapes in the resulting tiling are all the same as the edge lengths in the original tiling, and such that the angles between edges connected to the same vertex are multiples of τ / 10. Admittedly, in Figure 4e of [9], there is one virus proposed as a T-K tiling that does not preserve this property; but we note that that virus is more successfully modelled as a C-K tiling [20].

We call T-K tilings "spherical Penrose tilings" for several reasons. Firstly, they preserve at least a few aspects of the Penrose tiling, having global five-fold symmetries, and can have local ones too. Secondly, T-K tilings that resemble arbitrarily large Penrose tilings can easily be made with a large enough choice of R. And thirdly, the consistent angle sizes and edge lengths give the T-K tilings a great deal of resemblance to the various versions of the Penrose tiling.

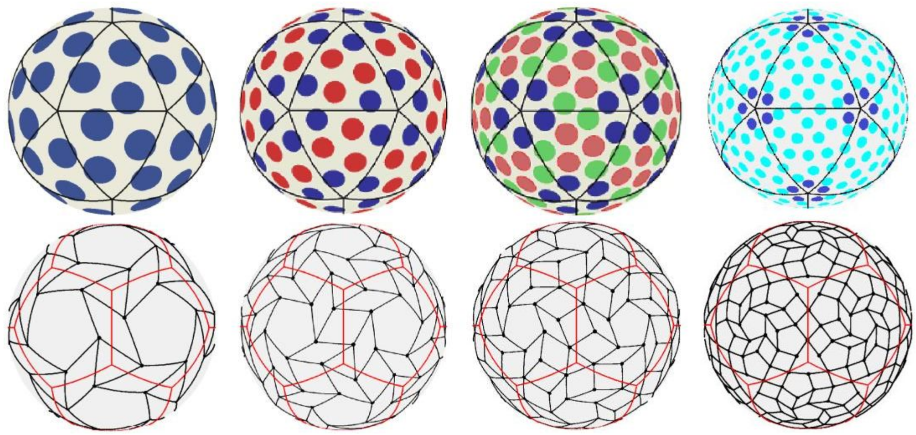

Figure 5: *Top: protein positions for each of the viruses seen in Figure 2 indicated with colored dots on spherical surfaces. Below: other T-K tilings that have vertices at the same positions as those colored dots, and edges at the locations of protein interactions. Note that though there appear to be equilateral triangles among the T-K tilings, in actual fact their interior angles are not τ / 6 but τ / 5. From [9], used with permission.*

Three important aspects of the Penrose tiling are its substitution rules, it's tiles' "matching rules", and its aperiodicity, and we will now address them. Firstly regarding the substitution process, we have not formally investigated whether it can interact meaningfully with T-K tilings, but we conjecture that it could be made to work, as it is possible to do something similar with the hexagonal lattice in C-K tilings. Secondly we also suspect that violations of the matching rules could be constrained to the dodecahedron's threefold symmetries (Figure 6). And finally, T-K tilings are not aperiodic; but we would argue that they can still be called spherical Penrose tilings because the Penrose tiling's aperiodicity is not unique and so cannot be the only thing that makes the tiling uniquely interesting to people. Additionally, two famous purported uses of the Penrose tiling include the decorations on the Darb-e imam shrine [7] and a toilet paper design [5], which are actually both periodic. Both have been argued to be Penrose tilings for different mathematical reasons, in the latter case by Roger Penrose himself, as part of successful legal action.

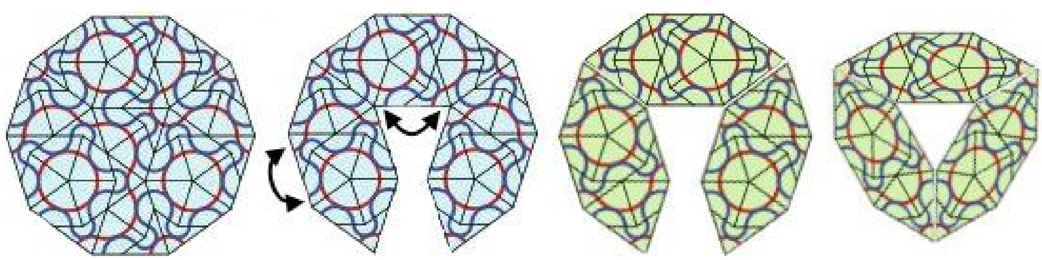

Figure 6: *Possible scheme for turning a region of a penrose tiling into a three-fold-symmetry with a localized violation of the matching rules (the central triangle). Modified from [7]*

An Algorithm for Generating a Subset of T-K Tilings

While creating the interactive demo described previously [16] we have implemented two algorithms which, together, generate some T-K tilings based on the Konevtsova Penrose-like lattice (Figure 4a), hereafter written as "T" and assumed to have edge-length 1. Algorithm I is a simple "visualizer", corresponding to Figure 4; a similar algorithm appears to have been used by others [8][19].

The input to algorithm I is T, together with a single point P_1 not at the origin, and its output is a mapping of part of T to $\mathbb{S}^2$. P_1 defines one corner of the pentagon that is to be cut out of the flat tiling. We define P_2 as the rotation of P_1 about the origin by $\tau / 5$. The triangle with corners P_1, P_2 and the origin now comprises a fundamental domain of our dodecahedral wrapping paper pattern. We search that fundamental domain for vertices of T, and map them to the copies of the fundamental domain (which can be thought of as the faces of the pentakis dodecahedron). To do this we simply use P_1, P_2, and their copies in the fundamental domain copies, as basis vectors.

Then, for every pair of mapped vertices, we check their distance on the surface of the dodecahedron. If their distance is equal to 1, we connect them with an edge. Our mapping is now obtained, see Figure 8. This algorithm runs in $O(n^4)$ for P_1 of length n. In our interactive demo we implemented a more convoluted algorithm that runs in $O(n^2)$, but this was not necessary as for all currently-known T-K tiled viruses, there are no more than 6 vertices in R.

Algorithm I, for arbitrary P_1, does not produce very satisfying outputs, as there may be points connected to only one other point; in other words, the output is not necessarily a "tiling." It also does not necessarily have the internal-angles property of the T-K tiled viruses.

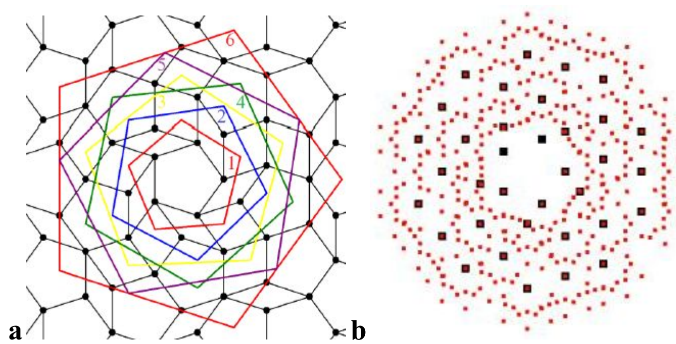

Figure 7: *Depictions of subsets of Z.* ***a****: the different states that obtain T-K points that model viruses, from [9], used with permission;* ***b****: our Z points (small) and also the lattice points (large)*

What we would like is a set Z of values for P_1 that preserves the property that all the polygons that make up the pattern have interior angles that are integer multiples of $\tau/10$. The purpose of algorithm II is to provide this, but the task is non-trivial. We attempt to do so in the following way: let D be the set of unit vectors pointing to the vertices of a decagon whose center is at the origin, and let V be the set of vertices in T. Then:

$$\{v + \alpha * d \; : \; v \in V,\; d \in D,\; \alpha \in \{0,\; \varphi - 1,\; 1,\; tan(\,\tau / 10\,)\}\} \subset Z$$

The reasoning behind the method (including the rather arbitrary-sounding values for α) is as follows: the point P_1 will end up lying on a threefold symmetry of the spherical tiling that the algorithm outputs. Therefore P_1 will either be on a vertex of the output, or it will be at the center of a shape with threefold rotational symmetry. To get points of Z that correspond to the former case we simply need to include all the vertices of the tiling in our set ($\alpha = 0$). To get points corresponding to the latter case we need to consider all threefold-symmetrical polygons with interior angles that are integer multiples of $\tau/10$, and then put P_1 at their centers. We have found this to be non-trivial, and so algorithm II is a compromise: it can output all T-K tilings currently known to model real viruses, and a few more, but we admit it almost certainly misses out on some beautiful T-K tilings, and is therefore not ideal. A method based on finding potential points of two-fold symmetry may have been better. Intriguingly it also produces outputs in which edges cross one another; we are not sure whether these should be considered valid outputs.

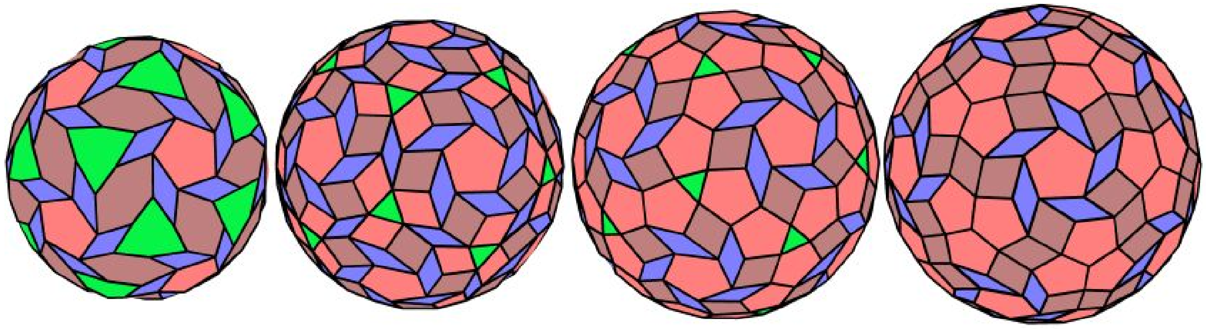

Figure 8: *Some outputs from the algorithm, which we hope will inspire some sculptures.*

T-K Tilings Outside Virology

Figure 9: *Existing artworks that are T-K tilings:* ***a****: Dome of the Jameh mosque in Isfahan, Iran[10];* ***b****: sculpture by Craig Kaplan [8] using a different algorithm to ours* ***c****: a lampshade that exactly—and accidentally—imitates the model for the structure of polyomavirus created by Reidun Twarock [3]* ***d****: Origami sculpture from Francesco De Comité, all used with permission. Note that all internal angles of all polygons in both tilings are multiples of* $\tau / 5$

T-K tilings have already been explored in mathematical sculpture [8][19][2]. We propose that the works in Figure 9 should be considered T-K tilings. One of them is a dome that was created by medieval Islamic artists several hundred years ago. We note that it has spots that are essentially threefold symmetries; we suggest that their knowledge of dodecahedral tilings may have been used in its construction.

There are many more possibilities for sculptures to be based on highly symmetrical wrapping paper patterns. We direct interested sculptors to another paper by Olga Konevtsova [13] in which a 12-fold symmetrical quasilattice is wrapped around the surface of a snub cube, obtaining beautiful and unique patterns. The general scheme to be investigated is one where a wrapping paper pattern is created by partnering a lattice with a polyhedron chosen for the "angular defect" at the polyhedron's vertices. The C-K tiling is created by partnering the 6-fold hexagonal tiling with an icosahedron, whose angular defect is τ/6; the T-K tiling is created by partnering the 5-fold Penrose tiling with a dodecahedron, whose angular defect is τ/10; the snub cube's angular defect is τ/12, and it is partnered with a 12-fold lattice.

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