On a Better Golden Rectangle (That Is Not 61.8033. . .% Useless!)

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Abstract

In which (A) it is alliteratively alleged that the $\phi \times 1$ Golden Rectangle™ is a ϕction of self-referential spatial elegance, unworthy of the reverent ϕxation with which ϕlosophers of beauty have for too long worshipped it; and (B) it is deϕnitively demonstrated that the $\sqrt{\phi} \times 1$ rectangle is a far more satisϕing, lean, golden mean, æsthetic ϕst-ϕghting machine. We explore its recursive subdivision into self-tilings exhibiting emergent global structure and patterns, and show its relationship both to golden ratio-based identities and to Ammann's bee tile, with a little art phistory added.

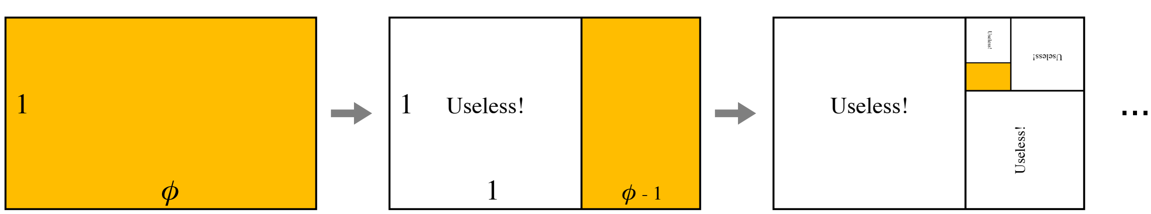
Complaint

For centuries, indeed millennia, smart people who should know better have ex-, de-, pro- and generally -claimed that the Golden Rectangle™ is fabulous, and wonderful, and oh-so-superbly proportioned, the very ephitome[1] of framed elegance among all things artfully rectangular, most worthy of all manner of æsthetic reference, preference, and deference, yada, yada, yada.

Well, it's time to put a stop to this facile, faithful, unfulϕlling nonsense. Of all things auric, the Golden Rectangle™ is—in my not so humble ophinion—far and away just about the *worst* example of "wrecked-angular" elegance in the parthen. . . er, pantheon of ϕ-dom. In short, yes, I'm sorry to say, the over-marketed, putatively sacred Golden Rectangle™ is a mathemæsthetic object unϕt for the burden of elegance and phiness(e) so worshipfully and steamingly heaped upon the poor thing.

As an Allegedly Self-Referential Object, the Golden Rectangle™ Is Nearly 62% Useless!

Phigure 1, left, shows yet another Miserable Golden Rectangle™ (hereinafter MGR) in its customarily ordained, "canonical" (ahem) wider-than-high form. Its dimensions are either $1 \times \phi$, where $\phi = (-1 + \sqrt{5})/2 = 0.618033\ldots$, or $\phi \times 1$, where $\phi = (1 + \sqrt{5})/2 = 1.618033\ldots$. Choose your $x^2 - x - 1 = 0$ poison; here we will opt for the larger (by) one ($\phi = 1.618033\ldots$).



Phigure 1: *The usual MGR subdivision accumulates self-referentially useless, non-Golden squares.*

The salient, semi-self-reverential property that everyone always *ooh*s and *aah*s about is this: if you carve off a square of side 1 from the MGR, you are left with another smaller $1 \times \frac{1}{\phi}$ MGR,

[1]Yes, my 'h's and/or 'p's can be silent. For non-English-speaking ϕlomorphs, feel free to sound-substitute "phi" or "fi" (as appropriate) for 'ϕ'.

rotated 90° (Phigure 1, center). And hence, one can perform another carving on the smaller MGR, iterating *ad inϕnitum* (Phigure 1, right), without drifting away from the same, simply divine proportion, which drifting otherwise occurs when using a greater or (judgmentally speaking) *lesser* ratio. From there—assuming one makes the appropriate binary symmetry choice forever—the usual true discussion of logarithmic spirals and false discussion of nautilus shells [3] ensues. Yawn.

Although the MGR is the only rectangle for which this goldenectomy holds—thereby selϕshly drawing much unwarranted attention to itself—the blatant elegance problem is that, after dividing the MGR into two pieces, the smaller piece (just under $2/5$ of the original) is the only part that is still golden; the majority (just over $3/5$) is just ... a flippin' (i.e., symmetric) unit square! But the unit (or any other) square has *absolutely nothing* golden about it to recommend itself. Nada. $0.\overline{0000000}$. Zip. Zilch. $\frac{7}{\aleph_0}$. 0^{41}. The so-called "empty set." $\bigcup_{i=0}^{\frac{\infty}{0}} 0 \times 0$; *see, e.g.*, lost causes.

It's like being presented with a piece of meat, billed as a ϕne steak, but which is fully $1/\phi = 61.8\ldots\%$ inedible fat, to be discarded (or fed to ϕdo) by those of us who are the more discerning ephicures! Any ϕnagling butcher foisting this scam off onto his customers would be rightfully subject to complaint, maybe even Madofϕan conϕnement for a ϕnancial pyramid[2] scheme!

The $\phi \times 1$ rectangle is, then, only minimally golden-ish. Its boring and useless square *gnomon* (that which is left over after removing a self-similar piece) is a symptom of the MGR's inherent inelegance and diminished mathematical beauty. MGR is, quite literally (well, okay, numerately) an æsthetic misϕt, due to a basic ϕ-ological mistake: *Everyone's been shoehorning ϕ's one-dimensional wonder into the wrong two-dimensional rectangle!* Bear with me now in proving this.

Divi $\frac{n}{d}$ ing a Rectangle into Self-Similar Pieces

Putting wordplay, snark, sound puns, and typogra-phi-cal mischief temporarily aside, without loss of congeniality, consider an $r \times 1$ rectangle, where $r \geq 1$, shown wider than high. What value(s) of r permit this rectangle to be subdivided into exactly N similar sub-rectangles $R_1, \cdots, R_N$?

The trivial answer, for $N=1$ and $r=1$, says that a square is similar to itself. The obvious non-trivial answer, for $N=4$, is also $r=1$, i.e., a square can be subdivided into four similar sub-squares. And indeed, for $N=k^2$, any rectangular r works. But what values of r work for $N=2$ or $N=3$?

The well-known solution for $N=2$ is the $\sqrt{2} \times 1$ rectangle, which can be halved into two similar sub-rectangles, each rotated 90°. This is the basis of European paper sizes, where, for instance, a sheet of paper of size A3 can be cut into exactly two sheets of size A4. Subdivide just one of those two sub-rectangles and, by induction, the $\sqrt{2} \times 1$ rectangle will work for any N.

As shown in Figure 2, when $N=3$ there are three distinct (disregarding symmetry) solutions. With a bit of algebra, one finds the three solutions are $r = \sqrt{2}$, $\sqrt{3}$, or ... $\sqrt{\phi}$.

The third $\sqrt{\phi} \times 1 = 1.2720196\ldots \times 1$ solution, which we will call GR, is unexpectedly interesting, not just because of the appearance of $\sqrt{\text{the golden ratio } \phi}$, but also because each of the three similar sub-rectangles is a different size, accomplished using different orientations.

Let $r = \sqrt{\phi}$. Scale each of GR's side lengths by $r^4 = \phi^2$, so that while remaining similar it becomes an $r^5 \times r^4$ rectangle (Figure 3, center). Label the three component rectangles in the subdivision as A, B, and C, from largest to smallest. Sub-rectangle A is similar, so its short side is r^3 and its area is thus $r^4 r^3 = r^7$. The short side of sub-rectangle B must be $(r^5 - r^3) = r(r^4 - r^2)$. But recall that $\phi^2 - \phi = 1$, i.e., $r^4 - r^2 = 1$. So the short side of B simplifies to just r, which

[2] Do not—I repeat, do not!—get me started on MGR and the pyramids, or I will need a deϕbrillator!

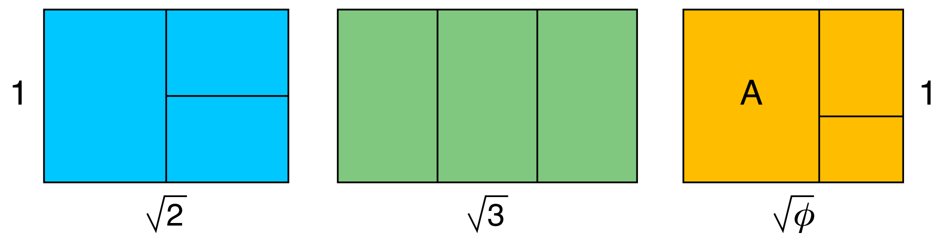


Figure 2: *Three distinct solutions to dividing a rectangle into three self-similar smaller rectangles.*

by similarity implies that B's long side is r^2. But that "in turn" means that sub-rectangle C must be $r^1 \times r^0 = r \times 1$. So we see that $r^5 = r^3 + r$ (horizontal lengths) and that $r^4 = r^2 + 1$ (vertical lengths), both of which are the same golden ratio identity as $\phi^2 = \phi + 1$. And summing areas, we get $r^9 = r^7 + r^3 + r$, which upon dividing by r is equivalent to the related identity $\phi^4 = \phi^3 + \phi + 1$.

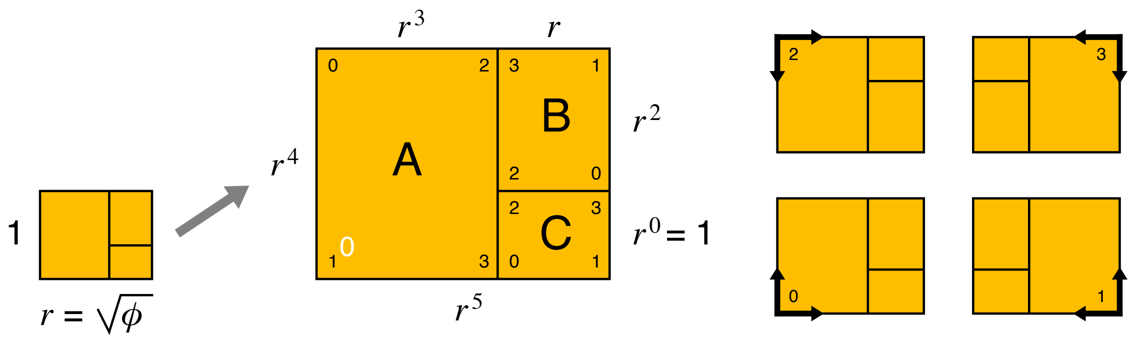


Figure 3: *Left: The subdivided* $\sqrt{\phi} \times 1$ *rectangle. Middle: Scaled by* $r^4 = \phi^2$*. Right: Each of A, B, and C has four possible coordinate systems based on which corner is treated as an origin.*

So GR gets to be goldenly recursive *everywhere*—no ugly leftover squares that infinitely sum to gnomonic uselessness with area $\sum_{i=0}^{\infty} \phi^{-i} = \frac{1}{\phi - 1} = \phi \times 1 =$ MGR (à la Phigure 1, right). QED.

Recursive Subdivision Everywhere Down to a Constant Level

Using translation, mirroring, and rotation, any GR has four possible coordinate systems by which to anchor a subdivision, where each corner can be an origin, labeled 0–3 (Figure 3, center and right). Therefore, there are $4^3 = 64$ ways to assign coordinate systems to the three component sub-rectangles A, B, and C, given the current coordinate system of their parent GR. Once a coordinate system is chosen for each of A, B, and C, one can recursively carve each into a next level of sub-sub-rectangles, which composes the transformations iteratively down to some level. Because the subdivision into three self-similar rectangles is asymmetric, the 64 possibilities have distinct forms.

The results are visually fascinating: wispy, self-similar cloud-like patterns emerge at many scales, due to how the tree of linear transformations brings the largest (lightest) As and the smallest (darkest) Cs together spatially in various ways. For a given recursive limit down n levels of A, B, and C, the result for each is a self-similar tapestry of 3^n GRs, in a variety of different sizes and orientations. Figure 4 shows four such recursive subdivisions for $n = 9$ (each rotated 90° to better fit on the page). Each features distinct, emergent self-similar patterns created by the exact same set of $3^9 = 19683$ GRs. When using the same line width to draw subdivision lines, smaller rectangles appear darker or merge. Global structure emerges from local texture. Is this not *way* more beautiful than one logarithmic spiral, which is self-similar at only one measly 0-dimensional point?

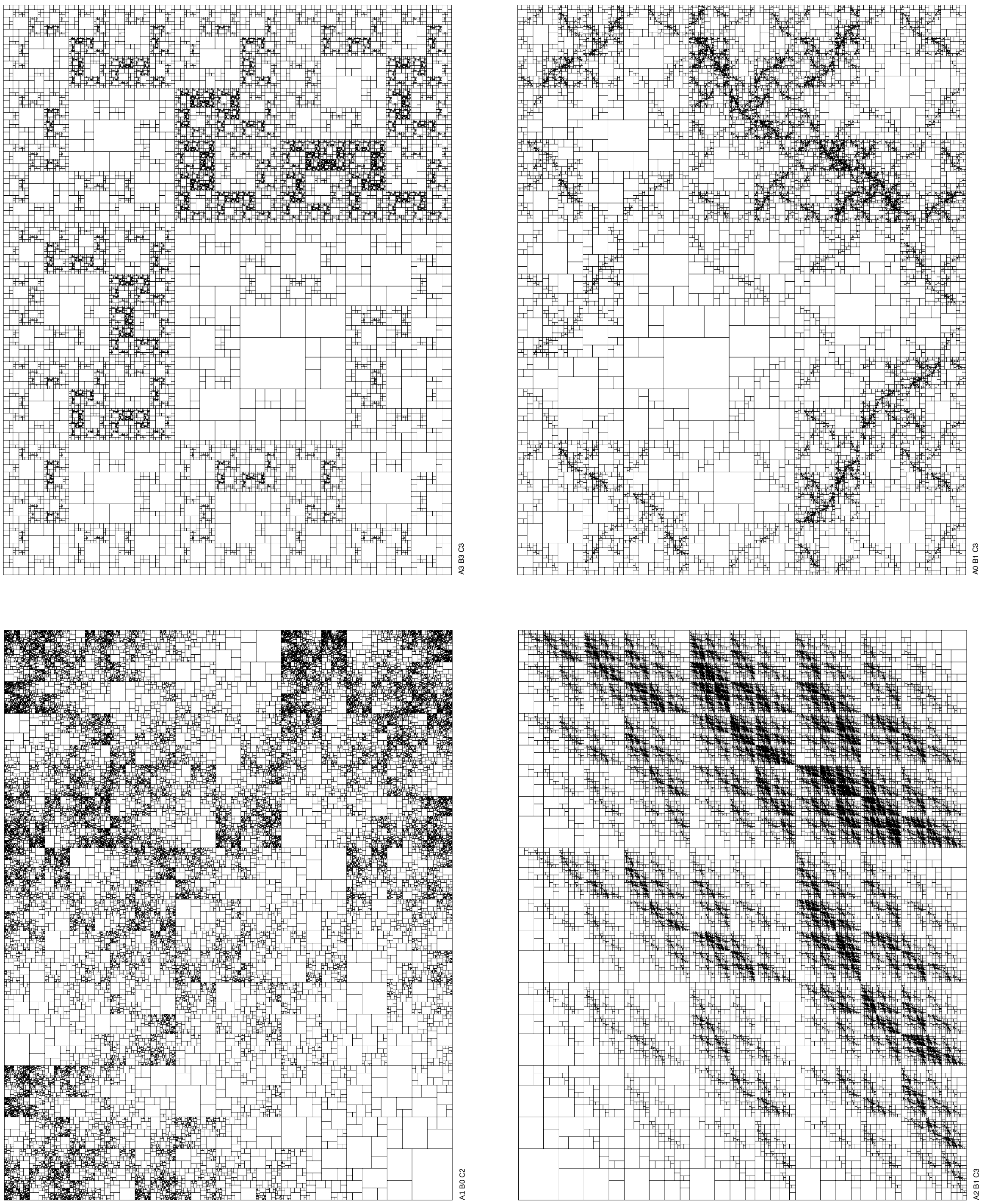


Figure 4: *Four of the 64 subdivision symmetries, each $n = 9$ levels deep. By construction, each uses the exact same set of 3^9 = 19683 sub-tiles. Top left: (A3 B3 C3) A new set of GRs emerges along rotated lines. Top right: (A0 B1 C3) The smallest GRs coalesce into more sinuous patterns. Bottom left: (A1 B0 C2) Seemingly chaotic swirls emanating from one corner. Bottom right: (A2 B1 C3) A self-referential diagonal sweep.*

Subdivision Lags Result in Aperiodic Tilings Using Only Four Sizes of GR

The golden ratio is a special constant because it represents an exact balancing point between additive and multiplicative growth. Hence, a recursive subdivision of A down more levels than of smaller B, and of B down a little more than of smaller C, yields only a small set of sub-rectangles exactly comparable in size. So at *each* recursive level n, subdivide A to level n, B to level $n-2$, and C to level $n-3$. These lags counterbalance the decreasing exponents on the right side of the area identity $\phi^4 = \phi^3 + \phi^1 + \phi^0$. Each of the 64 "size-limited" results is a tiling of GR with smaller copies of itself. But only four different tile sizes are produced. If, for each starting level n, GR is scaled by ϕ, every such tiling for any given n then represents an identity that defines ϕ^n as a sum of the first four powers of ϕ, i.e., $\phi^n = a\phi^3 + b\phi^2 + c\phi + d$, where the integer values of a, b, c, d count the number of tiles of each of the four size classes, in order of decreasing area. By construction, each of the four lag subdivisions in Figures 5 and 6 comprises the exact same set of tiles. In Figure 6, right, one can easily verify that the lag subdivision of a GR with area $r^{19} \times r^{18} = r^{37}$ contains $(a+b+c+d) = (714+442+714+441)$ GRs of areas r^7, r^5, r^3, and r, respectively. This corresponds to the identity $\phi^{(37-1)/2} = \phi^{18} = 714\phi^3 + 442\phi^2 + 714\phi + 441$, which simplifies (using $\phi^2 = \phi + 1$) to $\phi^{18} = 2584\phi + 1597$. That in turn is an instance of the general identity $\phi^n = F_n\phi + F_{n-1}$, in this case using the Fibonacci number pair $(F_{18}, F_{17}) = (2584, 1597)$.[3]

The emergent global structures in these self-tilings are also unexpected, beautiful, intriguing, and can be remarkably different from one another. But because of the much more homogeneous collection of tile sizes, it becomes quite difficult to perceive the original A, B, and C, or otherwise discern the rule by which tiles are placed. Figures 5 and 6 show four examples of non-periodic tiling patterns (on their sides) exhibiting large-scale structures, both "regular" and seemingly irregular.

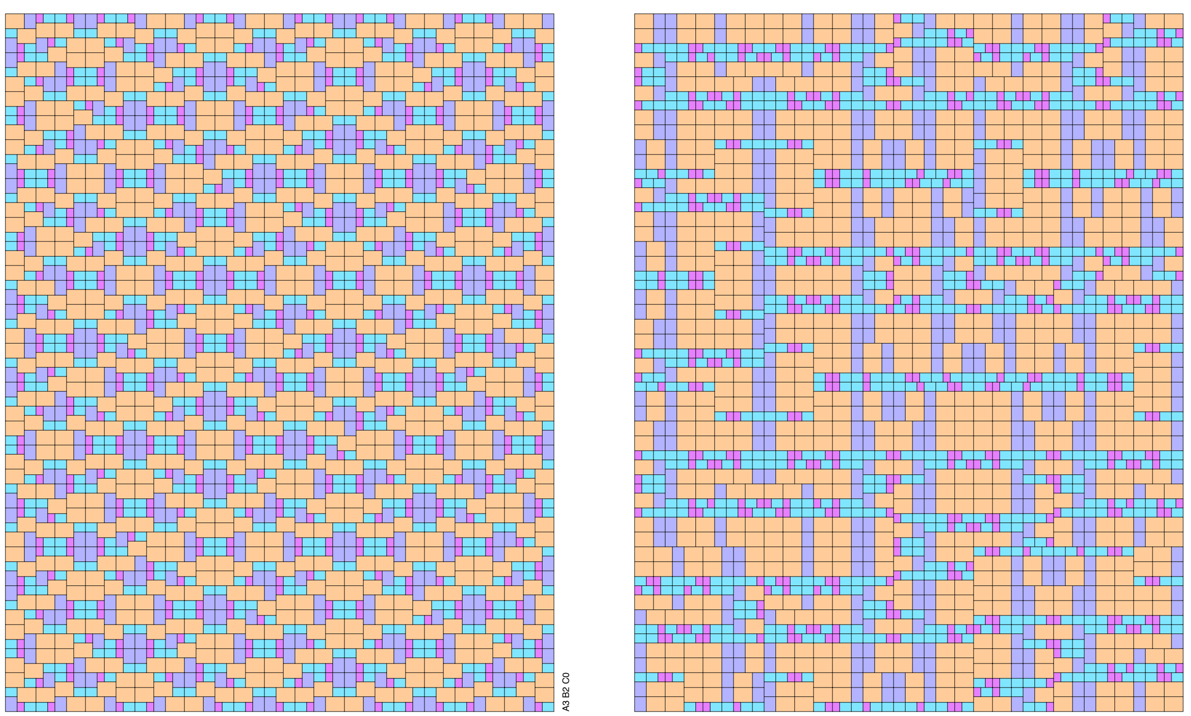


Figure 5: *Two lag subdivisions, using symmetry A3 B2 C0 (left) and A0 B2 C1 (right).*

[3] It's an interesting exercise to see what happens when you run n backwards for $n < 0$ in $\phi^n = F_n\phi + F_{n-1}$.

Especially intriguing about the two tilings in Figure 6 is that the large-scale emergent lines—whether diagonal or horizontal/vertical—occur in either pairs or as singletons, depending on the width of the gap between them. A relationship to the distribution of 0s and 1s in the binary Fibonacci word (sequence A003849 in [5]) would appear to be an excellent conjecture: the tree of linear transformations and subdivisions are likely isomorphic to the same production rule system. Figure 6, right, which I like to call a "Golden Plaid," exhibits the same structure horizontally as vertically. The stripes in one direction are larger than the stripes in the other by a factor of $r = \sqrt{\phi}$.

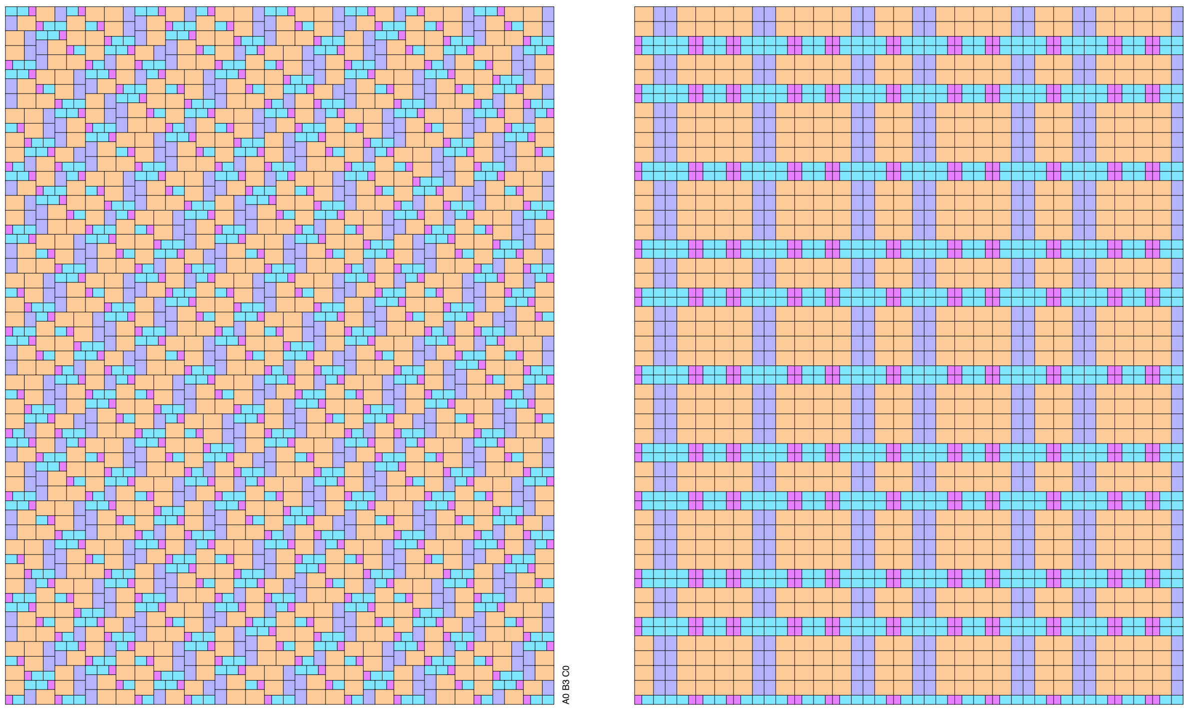


Figure 6: *Two more self-tilings. Left: Symmetry A0 B3 C0. Right: Symmetry A0 B2 C3 (a golden plaid!)*

Ammann's Aperiodic Golden Bee Tile

After carving GR up into A, B, and C, remove C. The result is also a self-similar figure that can be divided into just *two* different-sized copies of itself, each a mirror of the other, as in Figure 7, right. This shape is known as Ammann's "golden bee" tile [1]. With appropriate marking rules, just these two sizes (the larger is the smaller scaled by r) can tile the plane aperiodically [2].

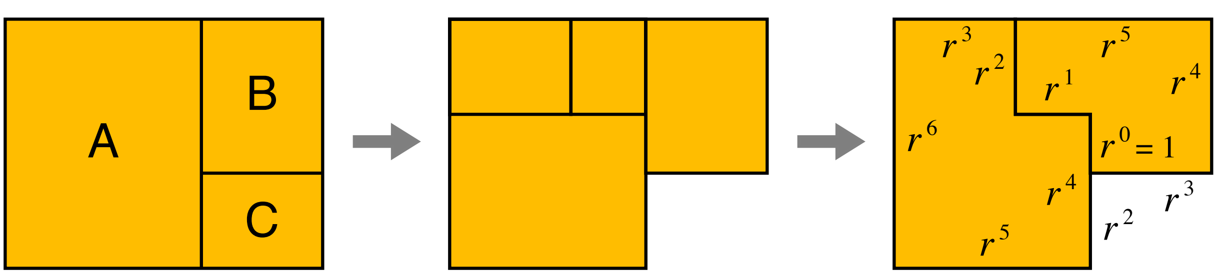


Figure 7: *Deriving Ammann's bee tile from GR.*

In 1977, Robert Ammann had been exploring aperiodic tilings built from various shaped tiles whose edges were based on powers of $\sqrt{\phi}$. His work was first described in 1986 in Grunbaum's *Tilings and Patterns* [2]. But some years before, some of Ammann's notes made their way via Martin Gardner to Benoît Mandelbrot, who in 1980 asked one of his illustrators (the present author) what to make of them. This is when I became aware of how awesome the $\sqrt{\phi} \times 1$ rectangle was.

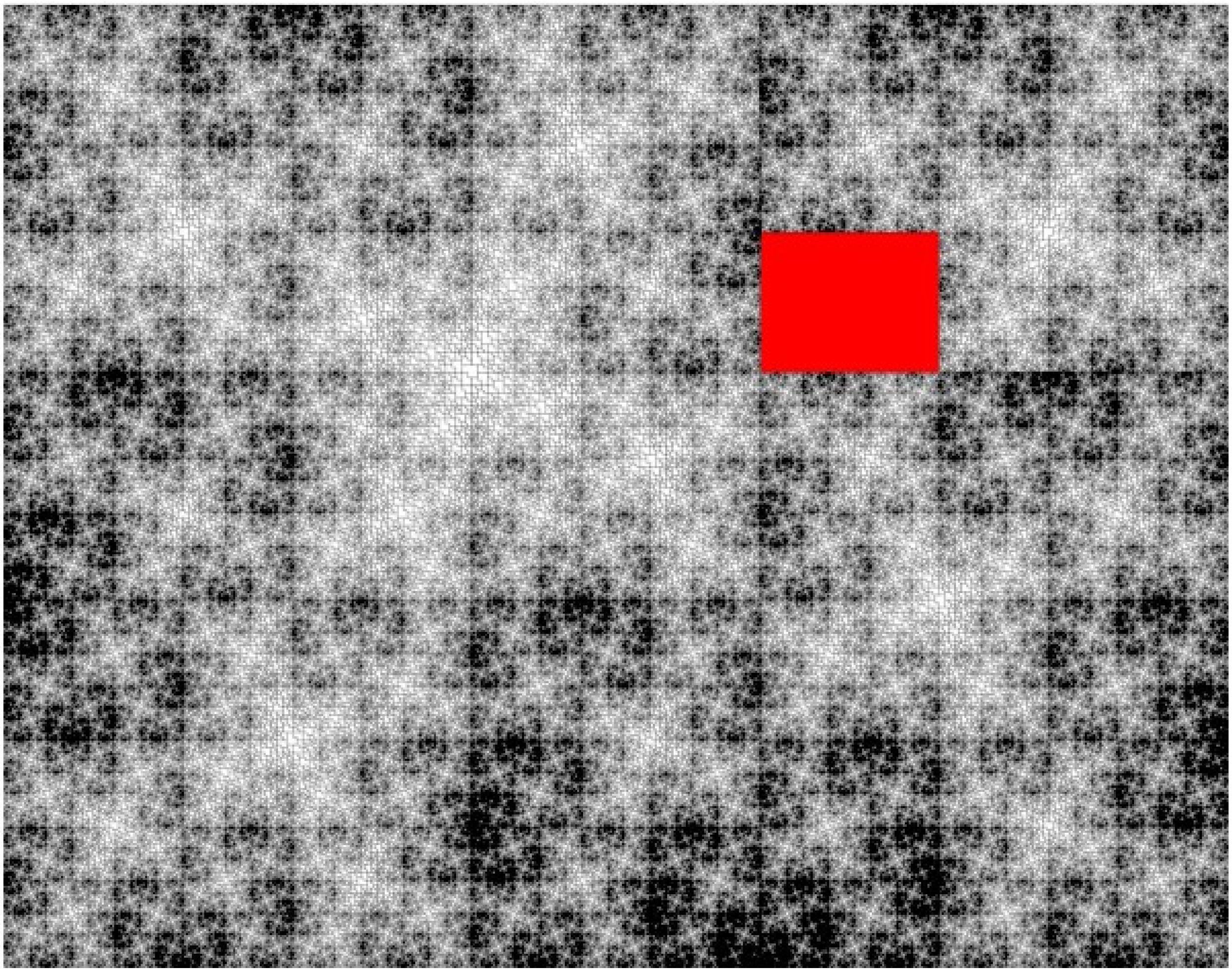


Figure 8: *"Golden Gnomon" shows myriad Ammann tiles, surrounding an unsubdivided GR.*

After going to a talk by Ammann a few years later (before he disappeared and died in obscurity), I explored and played with the bee's recursive subdivision. Based on techniques similar to those described above, an early mathematical art piece of mine, drawn in a pre-PostScript era on a pen plotter, entitled "Golden Gnomon" (Figure 8), was first exhibited at Yale University in 1985 [4].

Locally largest sub-tiles form diagonal "lines" of bee tiles along, e.g., the upper-left to lower-right diagonal of GR, and those line's similar analogs at smaller scales. By virtue of local symmetries, these lines implicitly create ghostly oval shapes of many sizes, creating a fascinating, almost tie-dye-like texture to the image. And, perhaps unsurprisingly, one can also find the Fibonacci numbers as an emergent property.

Looking at Figure 8, consider the set of horizontal bands, where the topmost band is bounded above by the line along the top of GR, and below by the line that passes along the bottom of the smaller unsubdivided GR (in the upper right of GR). These lines appear to cleave GR into two rectangles, but when you look carefully, they do not at locally lightest spots where a largest sub-tile finds itself. The height of each subsequent band (counting and measuring downwards) diminishes by a factor of $r^2 = \phi$. Two diagonal lines of locally largest golden bee tiles intersect at various points on the cleavage line between each band, where a locally largest Ammann tile interrupts the cleavage line otherwise dividing one band from the next. So the line across the top of the figure is completely straight; it has 0 interruptions. The next horizontal cleavage line down (by a factor of ϕ) has 1 interruption, as does the line below it. The next cleavage line below that has 2 interruptions, the next below has 3, then 5, then 8, then 13, etc.

Prayer for Relief

Call me a ϕnicky cynic, but the usual $\phi \times 1$ MGR plainly suffers from a golden identity crisis. It is worthy of at least 61.8033...% less attention than it regularly receives as an overly celebrated—yet minimally self-referential—geometric object. Whereas the more rephined $\sqrt{\phi} \times 1$ rectangle is a lean, golden mean, æsthetic ϕst-ϕghting machine. In both orthogonal directions it is sublimely and phierarchically suffused everywhere with Goldenness and Phibonacciness, evincing far more satisϕing aesthetic sensibilities, due to the invariably intriguing combination of self-similar regularity, aperiodicity, and asymmetry.

So to the inϕdel Miserable Golden Rectangle™, this aϕcionado most conϕdently shouts to phigh heaven: Φ on thee, you perphidious ϕgment of ϕine Platonic design! A new rectangle's in town! Your days of divinity are phinitely numbered, your phistory is ϕnally phinished!

I, for one, root for $\sqrt{\phi} \times 1$. Ammann.[4]

Acknowledgment

The amusing portions of the foregoing owe several high-ϕves to the Φresign Theater, whose surreal, artful, and deϕant "everything you know is wrong" aural antics and sound-of-words play long ago permanently disϕgured a signiϕcant "ratio" of the neural ϕbers within the author's unripened brain.

References

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[2] B. Grunbaum. *Tilings and Patterns*. W. H. Freeman, 1986, pp. 550–557.

[3] P. Gailiunas. "The Golden Spiral: The Genesis of a Misunderstanding." *Bridges Conference Proceedings*. Baltimore, Maryland, USA, 2015, pp. 159–166. http://archive.bridgesmathart.org/2015/bridges2015-159.html (as of 12/08/2022).

[4] E. Sigman. "The mathemæsthetics of fractals." *Yale Daily News*, Nov. 23, 1985. https://ydnhistorical.library.yale.edu/?a=d&d=YDN19851122-02.2.2 (as of 12/08/2022).

[5] N. A. J. Sloane. *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org/A003849 (as of 12/08/2022).

[4] This essay is for humorous and mathemæsthetic purposes only. The author disavows any support for any tenet of proportional divinity, sacred geometry, numerology, or other faith-based attempts to impart mysterious meaning to the myriad emergent phenomena of mathematical constraints whose beauties can only be understood from a position of enlightened and ratio-nal strength.

— Caphiat Emptor —