Sphairahedra and Three-Dimensional Fractals

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Abstract

In 2003, Kazushi Ahara and Yoshiaki Araki introduced sphairahedra, which are analogous to polyhedra but with spherical faces, in order to invent a new kind of fractal shape. Using inversions in faces of a sphairahedron, we can construct a tiling pattern of sphairahedra. In many cases, the boundary of the tiling converges to a three-dimensional fractal shape. Despite its interesting idea and impressive shape, there are few publications and computer graphics images. In this paper, we show variations of sphairahedra and their tiling patterns that were never visualized.

Introduction

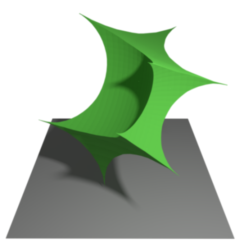
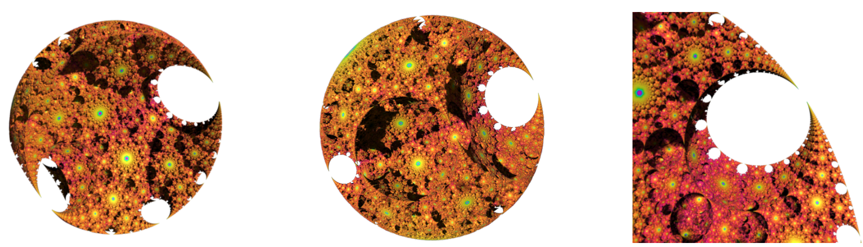


Figure 1: *Cube-type sphairahedron.*



(a) **(b)** **(c)**

Figure 2: *Images of a quasi-sphere rendered in different viewpoints.*

Kazushi Ahara and Yoshiaki Araki invented a new geometrical concept called a *sphairahedron* [1] in 2003. Sphairahedron is a coined word combining two words *sphaira-* (a prefix that means 'spherical') and *-hedron* (a suffix comes from 'polyhedron.') Mathematically a sphairahedron is analogous to polyhedron but with spherical faces. Figure 1 shows a cube-type sphairahedron. As we can see, each face of the cube is a part of a sphere.

We can make a tiling pattern of sphairahedra using inversion about their spherical faces. In many cases, the boundary of the tiling converges to a three-dimensional fractal shape as shown in Figure 2. The union of all the tiles is mostly homeomorphic to a three-dimensional ball. Thus, this boundary is called a *quasi-sphere*. Mathematically speaking, we consider a Coxeter-like group generated by the inversions in all of the spherical faces of the sphairahedron, and we obtain the tiling after transforming the original sphairahedron by each element of the group. The boundary of the tiling is called the *limit set* of the group, too. Also, under some technical conditions, the group is called a *quasi-fuchsian group* and the limit set a *quasi-fuchsian fractal*. The conditions are called ideality and rationality. When a sphairahedron is ideal and rational, we can tile it without gaps and self-intersections. We will introduce these properties in more detail later.

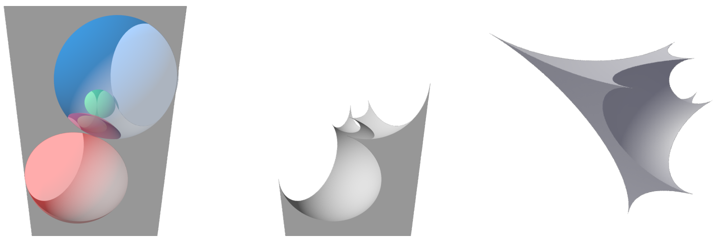
A quasi-fuchsian fractal is one of the three-dimensional fractals at an early era of visualizing fractals by computer. The video[1] posted by Ahara and Araki shows the fractal based on a cube-type sphairahedron in the case of only quasi-fuchsian. However, there are sphairahedra based on other types of polyhedra, and

[1]Quasi-fuchsian fractals: https://www.youtube.com/watch?v=3lcO9zRCv-4

we can allow tiles to self-intersect each other, that is the group is not quasi-fuchsian. In this way, we can see more varieties of fractal patterns than proposed by Ahara and Araki.

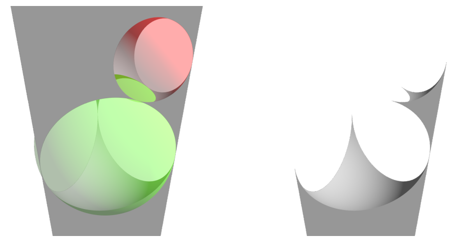
The three-dimensional tiling of hyperbolic polyhedra is well known. In comparison to this, sphairahedra and their tiling patterns are originated from four-dimensional hyperbolic geometry. The rise in dimension brings complexity and difficulty for visualization, but visualized shapes have impressive structures. In this paper, we will introduce a variety of sphairahedra and fractal shapes generated by them.

Sphairahedron



(a) *A* **(b)** *Infinite type* **(c)** *Finite type*

Figure 3: *Sphairahedron.*



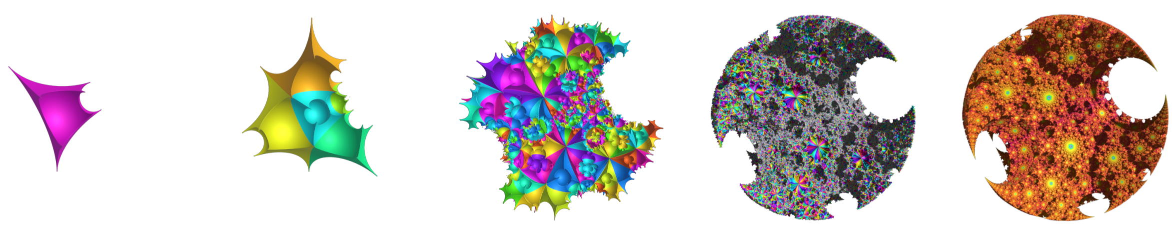
(a) *A* **(b)** *Infinite type*

Figure 4: *Semi-sphairahedron.*

First of all, we will describe the definition of a sphairahedron. Let $S^3 = R^3 \cup \{\infty\}$ be a three-dimensional sphere and let $\overline{D_1}$, $\overline{D_2}$, ..., $\overline{D_p}$ be some three-dimensional closed balls. We consider the complement A of the union of these balls, that is, $A = S^3 - (\overline{D_1} \cup \overline{D_2} \cup ... \cup \overline{D_p})$. If A is composed of simply-connected two components; in other words, A has two connected components and the first homology group of each component is trivial, we call one side of A a sphairahedron.

The image in Figure 3(a) is an example of A. We hollow out the S^3 with six balls: we remove three half space (three balls with infinite radius,) from S^3 and obtain the prism of infinite length, and we scoop out the prism by the remaining three-colored transparent balls as in Figure 3(a). A is composed of two parts, and each of the components is simply connected. Since it has six faces, and these faces are arranged as those of faces of a cube, it is called a cube-type sphairahedron. Especially, it is also called an *infinite type sphairahedron*, because one of the vertices of the sphairahedron is at the infinity. Similarly, the shape hollowed out by six finite balls in Figure 3(c) is called a *finite type sphairahedron*.

Moreover, we can loosen the definition of sphairahedron, that is, the case A has simply connected three or more components. Figure 4(a) shows an example of A with simply connected three components. Since it is the S^3 scooped out by five balls, this is a pentahedral prism type sphairahedron. We divide A so that one part of A has five faces as shown in Figure 4(b). We can regard the resulting shape as a singular case of a pentahedron, and we call it a *semi-sphairahedron*.

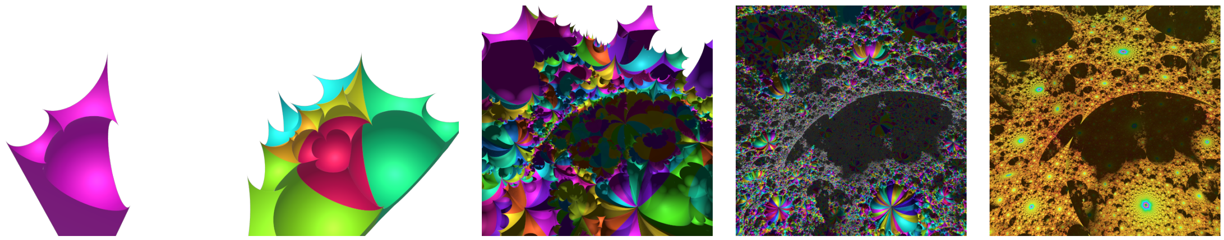


(a) *Step 1* **(b)** *Step 2* **(c)** *Step 5* **(d)** *Step 10* **(e)** *Final image*

Figure 5: *Tiling of a finite cube-type sphairahedron.*

Construct Fractal

In this section, we will show how to construct tiling patterns of sphairahedra. Figure 5 shows a process of the tiling of a cube-type sphairahedron. The sphairahedron presented in Figure 5(a) has six faces. We apply inversions in each spherical face to the original sphairahedron, and we get new six sphairahedra surrounding the sphairahedron in Figure 5(b). Next, we apply inversions in each of the new faces to the new sphairahedra, and we obtain more sphairahedra. We continue iterating inversions, and finally, we obtain a three-dimensional fractal shape as presented in Figure 5(e).



(a) *Step 1* **(b)** *Step 2* **(c)** *Step 5* **(d)** *Step 10* **(e)** *Final image*

Figure 6: *Tiling of a cube-type infinite sphairahedron.*

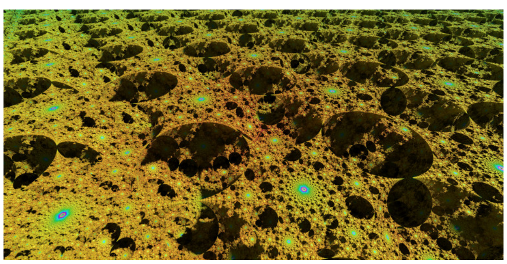


Figure 7: *Fractal terrain besed on the infinite sphairahedron in Figure 6.*

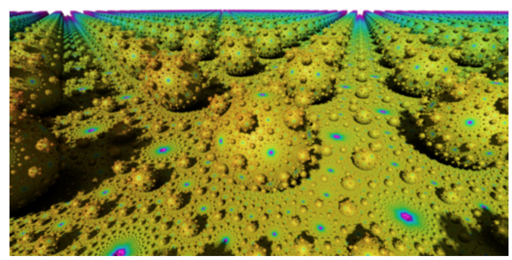


Figure 8: *Fractal terrain based on the infinite semi-sphairahedron in Figure 4.*

Also, an infinite type sphairahedron can be tiled as presented in Figure 6. The pattern converges to fractal terrain owing to reflections over side faces of the sphairahedron as shown in Figure 7. In the fractal terrain, we can find symmetry easily. For example, we can see hexagram-like terrain patterns in Figure 7. These patterns are originated from the dihedral angles of the side faces of $\pi/3$.

In the same way as the tiling of the sphairahedron, a semi-sphairahedron can be tiled by the inversions about its faces. The resulting fractal of the semi-sphairahedron is different from normal sphairahedron's one. Figure 8 shows the pattern generated by an infinite semi-sphairahedron shown in Figure 4. It is the union of an infinite number of balls circumscribing each other and no longer homeomorphic to a three-dimensional ball. Thus, It is not a quasi-sphere or a quasi-fuchsian. We will describe more about a tiling pattern of a semi-sphairahedron later.

In the images of fractals within this paper, each tile of the sphairahedra is colored according to the number of inversions. We use the color wheel to determine their color, and the tile's color varies in order of red, yellow, green, and blue. In Figure 5 (a) ~ (d) and Figure 6 (a) ~ (d), we refer color wheel with large steps to visualize each tile clearly. On the other hand, in the other images, we refer the wheel with smaller steps, and we find lots of tiles with many inversions in the blue parts of the fractal. We can also find that the blue parts themselves form the constant patterns.

In order to visualize these fractal objects using a computer, we have to compute an infinite number of sphairahedra. It often takes too much time. However, there are algorithms to render this kind of three-dimensional fractals efficiently. We use *sphere tracing* and *Iterated Inversion System (IIS)*. Sphere tracing is a kind of ray tracing techniques to render implicit surfaces. IIS is an algorithm introduced in our previous works, and it is used to visualize fractals originated from reflections or inversions. For more details about rendering algorithm, read our previous paper [2].

Up to this point, We showed tiling patterns without gaps between the tiles and intersections of the tiles. However, not every sphairahedra can generate such proper tiling patterns. To obtain them, we have to consider two mathematical properties of the original sphairahedron, that is, the sphairahedron should be ideal and rational. In the next section, we will introduce these properties.

Ideality and Rationality

We consider two properties to characterize a sphairahedron: ideality and rationality. First, we introduce ideality. Let P be a sphairahedron. We say P is *ideal* when all of the edges of P are mutually tangent at their vertices. A standard polyhedron, that is, a polyhedron with planar faces, never have this property. The second property is rationality. We say P is *rational* if each of the dihedral angles of the edges is equal to π/n for a natural number n.

For instance, all of the dihedral angles of the sphairahedron in Figure 3 are $\pi/3$. That is to say, all of the dihedral angles are expressed as π/n, and the sphairahedron is rational. Then, as presented in Figure 3(a), each of the vertices is the point of contact between three balls. Thus, the sphairahedron is ideal.

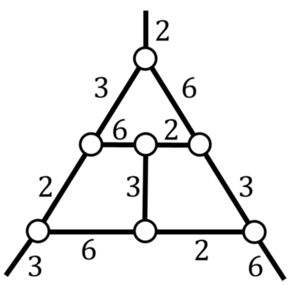


Figure 9: *Polyhedral Graph for infinite sphairahedron.*

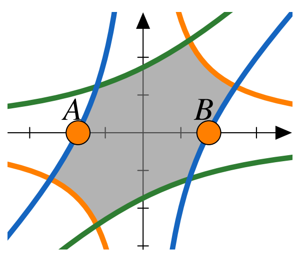
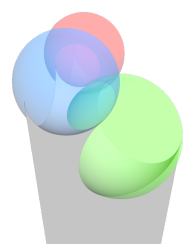
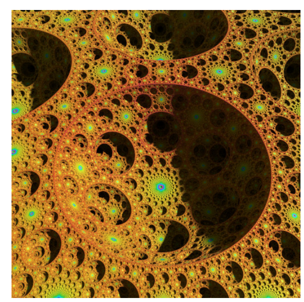


Figure 10: *Parameter space of the cube-type sphairahedron.*

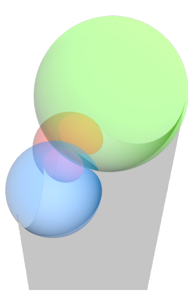


(a) *Sphairahedron*

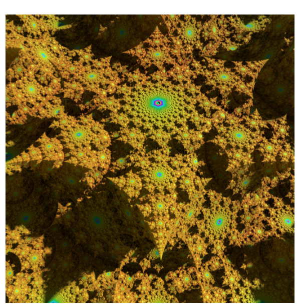


(b) *Limit set*

Figure 11: *Sphairahedron corresponding to the parameter A in Figure 10.*

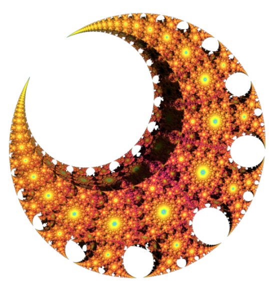


(a) *Sphairahedron*

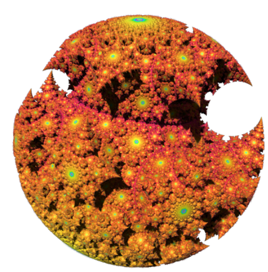


(b) *Limit set*

Figure 12: *Sphairahedron corresponding to the parameter B in Figure 10.*



(a) *Limit set*



(b) *Limit set*

Figure 13: *Limit sets based on the sphairahedron in Figure 12. Each of them is transformed by two different inversion spheres.*

Parameter Space of Ideal Rational Sphairahedron

Ahara and Araki worked on a classification problem of ideal rational sphairahedra and derived the parameter space of cube-type sphairahedra in their publication [1]. However, they showed only the limit set originated from a cube-type sphairahedron whose dihedral angles are $\pi/3$. In this section, we briefly show how to derive parametrization of ideal rational sphairahedra in general cases.

First of all, we have to determine the number of faces of a sphairahedron and how to arrange its edges between vertices. In order to represent a sphairahedron, we use a polyhedral graph as shown in Figure 9. The graph shows an infinite cube-type sphairahedron. The black lines represent edges, and the circles represent vertices. The three radial edges connect the infinite vertex and finite vertices. Each of the numbers beside the edges means a natural number n for the dihedral angle π/n.

Secondly, we enumerate the combinations of dihedral angles of each edge and choose one combination. In order to satisfy ideality, the sum of the dihedral angles at each vertex should be $(k-2)\pi$ for the number of the edges k connected to the vertex. Thus, the total of the dihedral angles at each vertex of the cube-type sphairahedron is π. Ahara and Araki found that the cube-type sphairahedron has seven combinations of the angles [1].

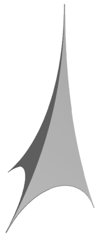
After we choose a combination of dihedral angles, we fix some side faces and height of some balls. In the cube-type sphairahedron, we fix side faces of the prism so that their interior angles are determined angles by the graph, and we fix the height of one of the balls to zero.

Finally, we parametrize the ideal rational sphairahedron respect to the heights of the rest of the balls. The positions and radii of the balls are decided on the basis of ideality and rationality in relation to the fixed prism and balls.

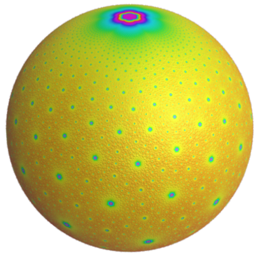
For example, we show the outline of the parameter space for the infinite cube-type sphairahedra whose all dihedral angles are $\pi/3$ in Figure 10. It is derived by Ahara and Araki in their paper [1]. Figure 11 and Figure 12 show sphairahedra and their limit sets corresponding to the points A and B on the parameter space. The coordinates of the parameter space represent the heights of the two balls for the sphairahedra in Figure 11(a) and Figure 12(a). The x-coordinate is the height of the green ball on the right side, and the y-coordinate is the height of the blue ball in front of the left side. The gray area surrounded by the three hyperbolas is the parameter space of ideal rational sphairahedra. In other words, if the parameter is contained in the gray area, the corresponding sphairahedron is ideal and rational.

It is ensured mathematically that the limit set of the sphairahedron is continuously deformed while the parameter varied in the parameter space. For instance, we find that there are many crater-like dents in the limit set in Figure 11(b). As we increase the height of the green ball on the right side, the dents rise, and we can see the hexagram-like terrain in Figure 12(b).

After we obtain an infinite sphairahedron, we can get finite sphairahedra from it. We fix a point which does not match vertices of the infinite sphairahedron, and we consider another sphere centered at the point. We call the new sphere an *inversion sphere*. We invert the infinite sphairahedron in the inversion sphere, and we get a finite sphairahedron. We can choose positions and radii of the inversion sphere, and the quasi-spheres are continuously deformed according to the configuration of the inversion sphere. In Figure 13, we show the two limit sets. Each of them is generated by the same sphairahedron in Figure 12 using two different inversion spheres. In this way, we can get many variations of sphairahedra and limit sets by choosing the inversion sphere besides we change the parameter.

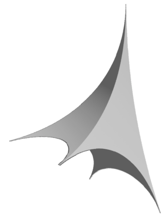


(a) *Sphairahedron*

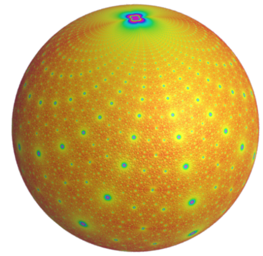


(b) *Limit set*

Figure 14: *Finite tetrahedron type.*

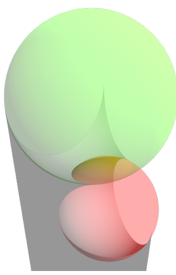


(a) *Sphairahedron*

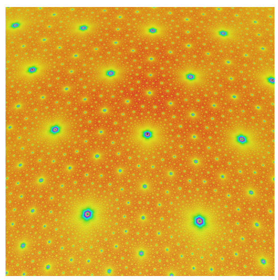


(b) *Limit set*

Figure 15: *Finite pentahedral pyramid type.*

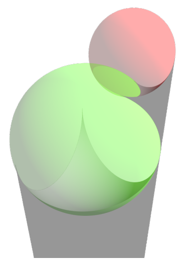


(a) *Sphairahedron*

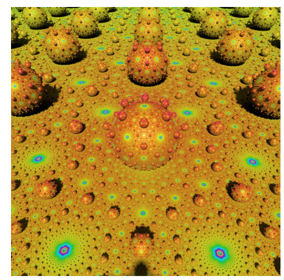


(b) *Limit set*

Figure 16: *Infinite pentahedral prism type.*

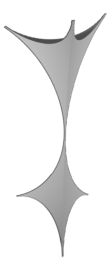


(a) *Sphairahedron*

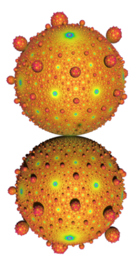


(b) *Limit set*

Figure 17: *Infinite pentahedral prism type with two components.*

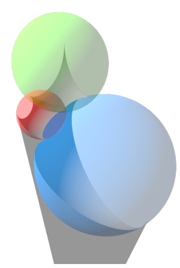


(a) *Sphairahedron*

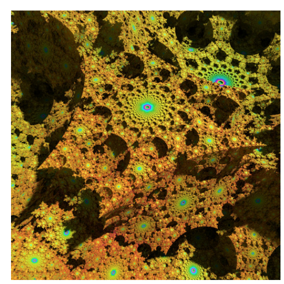


(b) *Limit set*

Figure 18: *Finite pentahedral prism type.*

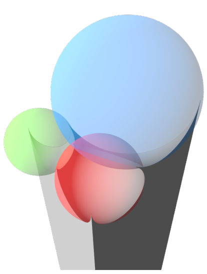


(a) *Sphairahedron*

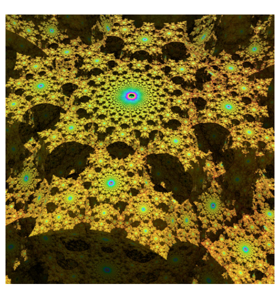


(b) *Limit set*

Figure 19: $(\pi/2,\ \pi/3,\ \pi/6)$ *infinite cube-type.*



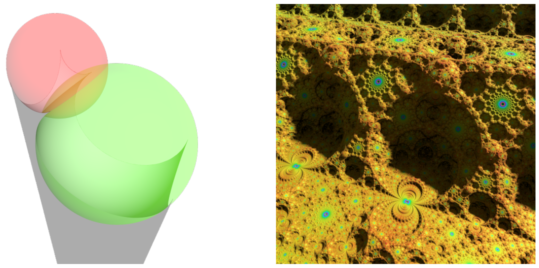
(a) *Sphairahedron*



(b) *Limit set*

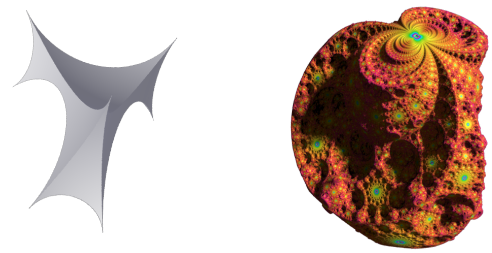
Figure 20: $(\pi/2,\ \pi/4,\ \pi/4)$ *infinite cube-type.*

Figure 21: *Cake-type hexahedron.*



(a) *Sphairahedron* **(b)** *Limit set*

Figure 22: *Infinite hexahedral cake-type.*



(a) *Sphairahedron* **(b)** *Limit set*

Figure 23: *Finite hexahedral cake-type.*

Variations of Sphairahedra

In the same way as the parametrization of the cube-type sphairahedron, we can obtain the parameter spaces of other types of the ideal rational sphairahedron. In this section, we will show some more examples of sphairahedra and their limit sets.

Figure 14 and Figure 15 show sphairahedra and their limit sets based on a tetrahedron and a pentahedral pyramid. Each of them has the unique combination of dihedral angles and a single parameter. Both of the limit sets are actual spheres, but their patterns of color are different from each other.

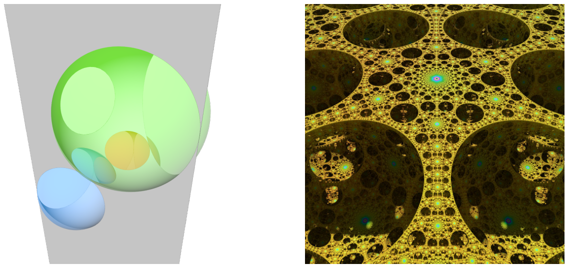
Sphairahedra based on pentahedral prism have six combinations of the dihedral angles. It turns out that most combinations of this type become semi-sphairahedra. The pentahedral prism type sphairahedra have one parameter for the height of the green ball on the left side in Figure 4(a). The limit sets of them have an interesting character. See Figure 16(a). It shows one of the semi-sphairahedron whose dihedral angles are $\pi/3$. The limit set shown in Figure 16(b) seemed to be a plane, but there is an infinite number of the spherical hollows under the plane. When we decrease the height of the green ball, the semi-sphairahedron becomes to be composed of two components as shown in Figure 17(a). The hollows rise, and they emerge on the plane as balls as presented in Figure 17(b). As described in the previous section, the limit sets of semi-sphairahedra are no longer quasi-spheres or quasi-fuchsian. Figure 18 shows finite type semi-sphairahedron and its limit set. It is easy to find that the limit set is composed of an infinite number of balls circumscribing each other.

We already introduced the cube-type sphairahedron whose every angle is $\pi/3$. According to the combinations of the angles, the patterns of the limit set are greatly changed. Figure 19 and Figure 20 show the sphairahedra whose dihedral angles of the side faces are $\pi/2$, $\pi/3$, and $\pi/6$, and $\pi/2$, $\pi/4$, and $\pi/4$ respectively. Each of the limit sets forms a crater-like shape. The shapes of the limit sets result from triangular reflections of side faces and difference in the height of the spherical faces of the sphairahedra.

Finally, we show another sphairahedron based on the hexahedron called cake-type in Figure 21. The infinite sphairahedron in Figure 22(a) have four side faces, and one parameter for the height of the smaller red ball on the left side. The limit set shown in Figure 22(b) has parallel translation symmetry along the vertical directions of the faces, and the difference between the heights of two spherical faces of the sphairahedron causes the difference of elevation of the terrain. A finite sphairahedron and its limit set are shown in Figure 23.

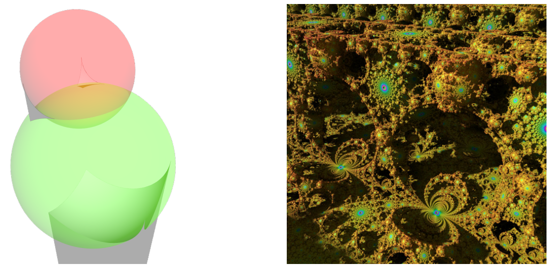
Breaking Ideality or Rationality

In the previous section, we dealt with ideal rational sphairahedra. If we increase the number of the faces of a sphairahedron more than six, it becomes difficult to obtain an ideal rational sphairahedron. The reasons are because the constraint of dihedral angles owing to rationality becomes more strict as the number of the faces increases and the number of the edges connected to one vertex is limited to at most four owing to ideality.



(a) *Sphairahedron* **(b)** *Limit set*

Figure 24: *cube-type sphairahedron corresponding to the outside of the parameter space in Figure 10.*



(a) *Sphairahedron* **(b)** *Limit set*

Figure 25: *Cake-type sphairahedron same type as Figure 22.*

In this way, we try to think about sphairahedra not always satisfying ideality and rationality. For instance, the parameter space in Figure 10 is a subset of the plane. In this section, we will examine sphairahedra and quasi-spheres corresponding to the parameter on the plane and outside of the parameter space. The sphairahedra may not be ideal or rational nor even sphairahedra, but their limit sets are meaningful shapes.

For example, we consider the cube-type sphairahedron in Figure 12. The limit set forms a hexagram-like shape, and we call six elongated parts of the hexagram arms. The arms get longer as the parameter approaches to the edge of the parameter space. When the parameter is at the edge (the point B in Figure 10,) the arms come into contact with neighboring arms. Then the parameter is outside of the parameter space, and the arms overlap each other as presented in Figure 24(b). In this case, the sphairahedron corresponding to the limit set may not be rational, and the other part of the sphairahedron gets a hole as shown in Figure 24(a).

Another example is in Figure 25. It is based on the cake-type sphairahedron in Figure 22. If we make the height of the smaller red ball too high, the resulting sphairahedron will be broken as shown in Figure 25(a). However, the resulting limit set in Figure 25(b) keeps the meaningful shape, and there are holes everywhere in the shape.

Summary

In this paper, we introduced many varieties of sphairahedra and their tiling patterns. The basic idea is not new, but the fractal terrain and semi-sphairahedra may be visualized for the first time. We also show sphairahedra which are breaking mathematical requirements. The limit sets have beautiful and attractive shapes, on the other hand, they also have many mathematical open problems. We may find a new problem or a clue to solve a problem from visualized images.

The first author is developing a web application to render sphairahedra and their limit sets interactively. The application, more visualized images, and three-dimensional models of sphairahedra are available at https://sphairahedron.net.

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