The Beauty of the Symmetric Sierpinski Relatives

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Abstract

The Sierpinski relatives form a fascinating class of fractals because they all possess the same fractal dimension but can have different topologies. The famous Sierpinski gasket is one of these relatives. There is a subclass consisting of symmetric relatives that are particularly beautiful. This paper presents an exploration of these relatives. These eight relatives are all fractal, however their convex hulls are polygons with at most eight sides. The convex hulls provide a way to tile the relatives to obtain other beautiful fractals. The fractals include gasket fractals and fractal frieze patterns.

Introduction

The Sierpinski Gasket is a well-known fractal, see Figure 1(a). It can be expressed as the union of three smaller versions of itself that have lengths scaled down by a factor of 2, so it has fractal dimension equal to $\ln 3 / \ln 2 \approx 1.585$. The class of fractals known as the Sierpinski relatives all possess the same scaling properties and hence the same fractal dimension. However, the topological properties of these relatives vary [7]. The gasket is multiply-connected (possesses holes). It is possible for a relative to be: completely disconnected (connected components consist of single points) as in Figure 1(b); disconnected with straight line segments (infinitely many connected components, components may be more than just single points because there are straight line segments) as in Figure 1(c); or simply-connected (no holes), as in Figure 1(d). The relatives are obtained by using the same scaling as the Sierpinski gasket and the symmetries of the square. Some relatives are symmetric about the diagonal from the lower left to upper right. This paper presents other fractals generated from these symmetric Sierpinski relatives and their convex hulls.

(a) (b) (c) (d)

Figure 1: *Sierpinski Gasket and three Sierpinski Relatives*.

Mathematical Background

One way to describe the Sierpinski gasket is as the unique attractor of an iterated function system (IFS) $\{g_1, g_2, g_3\}$. Let S be the unit square (vertices are (0,0), (1,0), (1,1) and (0,1)). The maps g_1, g_2, g_3 are contractive similarities (they preserve shape but scale down the size), that act on the unit square as in Figure 2. These maps all contract the lengths of the sides of the square by a factor of two and they all preserve the orientation of the square. The map g_2 moves the contracted square to the right by ½ while the map g_3 moves it up by ½. A relative of the gasket is the attractor of an IFS that consists of three maps that all map the unit square to a square with sides of length ½ and each of the maps may involve a symmetry transformation of the square.

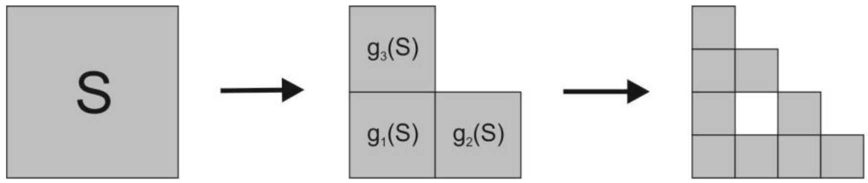


Figure 2: *The unit square and result of applying maps $g1$, $g2$, $g3$.*

Symmetries of the Square

There are eight symmetries of the square $\{a, b, c, d, e, f, g, h\}$: a (the identity, doesn't move the square); b (rotation by 90° counter-clockwise); c (rotation by 180° counter-clockwise); d (rotation by 270° counter-clockwise); e (reflection across the horizontal line through the center); f (vertical reflection); g (reflection across diagonal through lower left corner and upper right corner); and h (other diagonal reflection). The actions of the symmetries on a "J" placed in the unit square are shown in Figure 3.

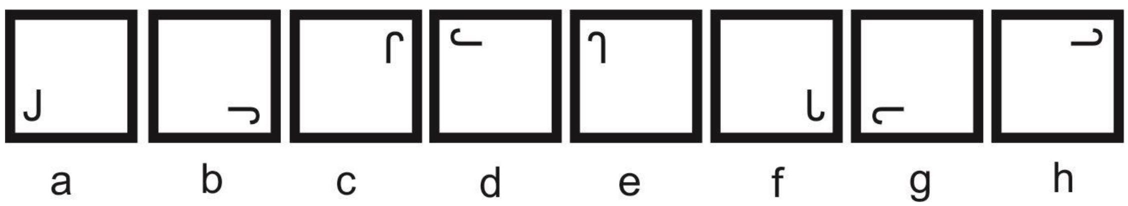


Figure 3: *Symmetries of the Square.*

A Sierpinski relative R_{xyz} can obtained from an IFS $\{f_1, f_2, f_3\}$ acting on the unit square S. The map f_1 involves contraction and the symmetry x; the map f_2 involves contraction, the symmetry y, and horizontal translation by ½; the map f_3 involves contraction, the symmetry z, and vertical translation by ½. Note that the contractions are all by a factor of ½, and each map takes the relative to a scaled down version of itself. Figure 4 displays the maps to obtain the relative R_{abd}. There are 8 choices for each symmetry of the square, thus there are $8 \times 8 \times 8 = 512$ possibilities. However, the resulting fractals are not all distinct. We will explain why there are 232 distinct Sierpinski relatives.

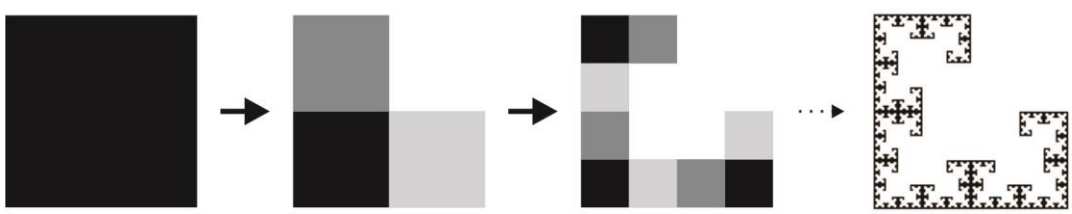


Figure 4: *Maps to obtain the relative R_{abd}.*

The Sierpinski gasket corresponds to R_{aaa}. Observe that the gasket is symmetric about the diagonal from the lower left to upper right (this is symmetry g of Figure 3). Due to this invariance under the map g, the gasket also corresponds to $R_{aag}, R_{aga}, R_{gaa}, R_{agg}, R_{gag}, R_{gga}$ and R_{ggg}.

Figure 5 displays eight relatives that are symmetric about the diagonal from the lower left to upper right: R_{aaa}, R_{abd}, R_{acc}, R_{adb}, R_{caa}, R_{cbd}, R_{ccc} and R_{cdb}. As with the gasket, this invariance under the map g means that each of the three maps in the IFS has a choice of two possible symmetries (the original from the list or the original composed with g). Each symmetric relative has $2 \times 2 \times 2 = 8$ choices for the IFS. For example, these are the choices that yield the same fractal as R_{abd}:

$$R_{abd}\colon R_{abe}, R_{afd}, R_{afe}, R_{gbd}, R_{gbe}, R_{gfd}, R_{gfe}.$$

There are 64 choices that yield symmetric relatives (it can be shown that these are the only symmetric relatives [7]). For the $512 - 64 = 448$ choices left for the IFS, each choice yields a relative that is congruent (via reflection across this diagonal from the lower left to upper right) to exactly one other choice. Hence there are 224 distinct fractals. In total, there are 8 symmetric plus 224 non-symmetric relatives for a total of 232. See [7,6] for more details.

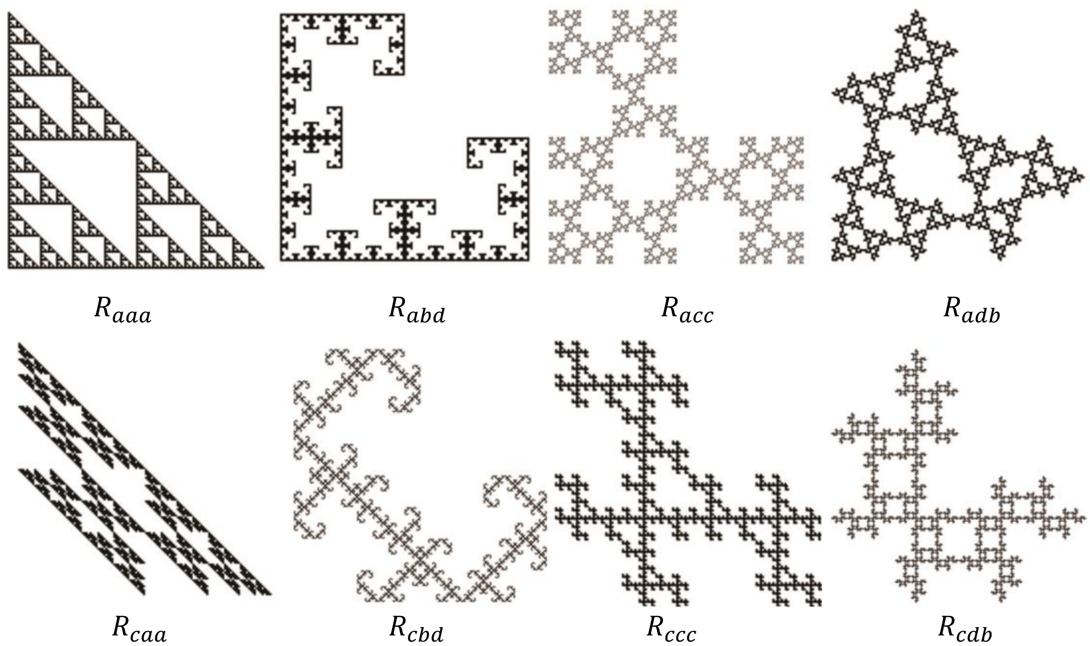


Figure 5: *Eight symmetric Sierpinski relatives.*

Convex Hulls

A set A is convex if for any two points p and q in the set, the line segment $\overline{pq}$ joining them is also in the set (Figure 6). The convex hull of a set is the smallest convex set that contains the set. One way to visualize the convex hull of a set of points is that it includes its boundary which is like an elastic band around the points, and it contains everything inside the elastic band. See [2,5] for more details.

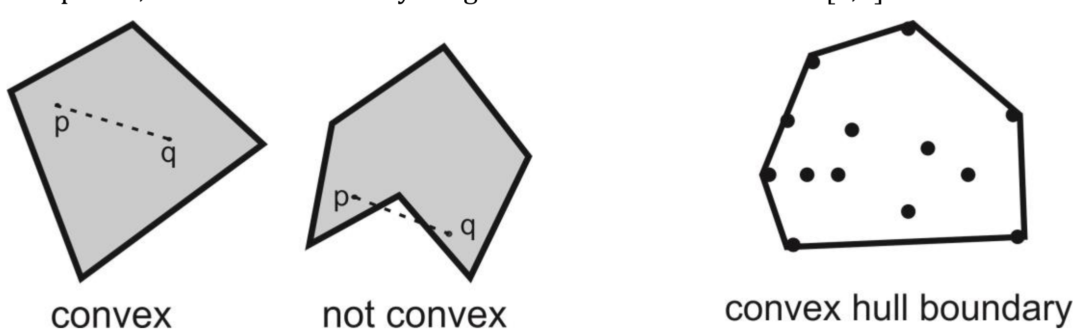


Figure 6: *Convex set and non-convex set; set of points along with boundary of its convex hull.*

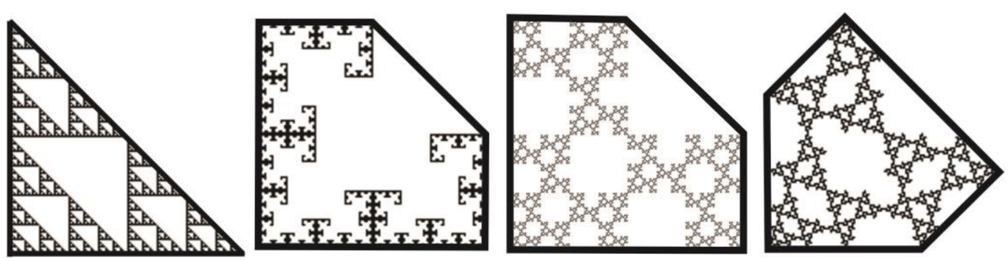
Convex hulls of the Sierpinski relatives can be a useful tool to help characterize and classify the fractals using topology instead of fractal dimension (work in progress, [8]). In general, the determination of the convex hull of IFS fractals can be quite complicated, see [9]. I am still working on a complete description of the convex hulls of the relatives. The symmetric relatives are much easier to work with. For some relatives, the boundary of their convex hull has vertices that are not on the unit square and this makes the determination more complicated. Figure 7(b) displays one such example, R_{bdg}. Consider the line segment between 2 points of R_{bdg} as in Figure 7(b). This line segment cannot be a side of the convex hull boundary, hence the convex hull boundary must contain vertices that are not on the unit square.

(a) (b)

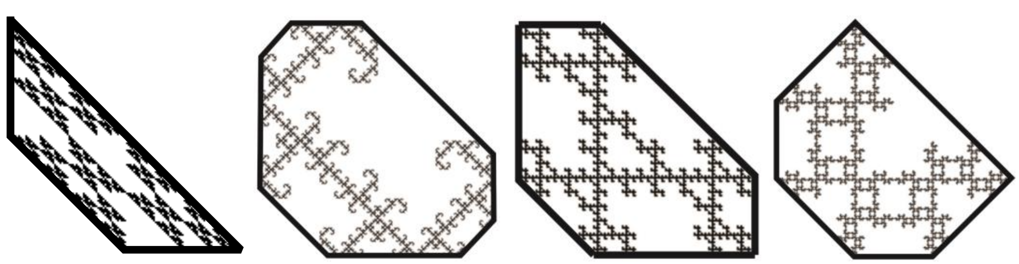
Figure 7: *R_{bdg} and line segment joining two points*

Convex Hulls of the Symmetric Sierpinski Relatives

One can show that the symmetric relatives all have convex hulls with polygonal boundaries, where the polygons have at most eight sides and have interior angles that are special angles (45°, 90° or 135°), see Figure 8. Mathematical details are forthcoming [8]. To describe the convex hull it suffices to give the vertices of the bounding polygon. These vertices are all on the boundary of the unit square S. Note that for a general relative, the convex hull may have a fractal curve boundary or a polygonal boundary and in many cases the vertices are not on the unit square. For the symmetric relatives, the convex hulls naturally fit together to tile new fractals or frieze patterns.



R_{aaa} R_{abd} R_{acc} R_{adb}



R_{caa} R_{cbd} R_{ccc} R_{cdb}

Figure 8: *Boundaries of the convex hulls of the symmetric Sierpinski relatives*.

For each symmetric relative, we describe the bounding polygon of their convex hull in terms of the vertices and the interior angles. We start with the vertex closest to (0, 0) and with $y = 0$, and go counter-clockwise around the polygon. The angles are listed in the same order.

R_{aaa} (the Sierpinski Gasket). Vertices: (0, 0), (1, 0), (0, 1). Angles: 90°, 45°, 45°.
R_{abd}. Vertices: (0, 0), (1, 0), (1, 1/2), (1/2, 1), (0, 1). Angles: 90°, 90°, 135°, 135°, 90°.
R_{acc}. Vertices: (0, 0), (1, 0), (1, 1/2), (1/2, 1), (0, 1). Angles: 90°, 90°, 135°, 135°, 90°.

R_{adb}. Vertices: (0, 0), (2/3, 0), (1, 1/3), (1/3, 1), (0, 2/3). Angles: 90°, 135°, 90°, 90°, 135°.

R_{caa}. Vertices: (1/2, 0), (1, 0), (0, 1), (0, 1/2). Angles: 135°, 45°, 45°, 135°.

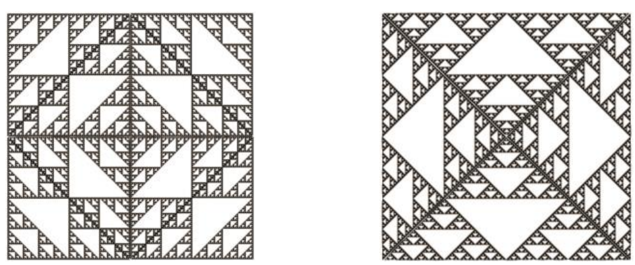
R_{cbd}. Vertices: (2/7, 0), (6/7, 0), (1, 1/7), (1, 3/7), (3/7, 1), (1/7, 1), (0, 6/7), (0, 2/7). All angles are 135° so there are many ways to tile this octagon.

R_{ccc}. Vertices: (1/3, 0), (1, 0), (1, 1/3), (1/3, 1), (0, 1). Angles are 135°, 90°, 135°, 135°, 90°, 135°.

R_{cdb}. Vertices: (1/3, 0), (2/3, 0), (1, 1/3), (1/3, 1), (0, 2/3) and (0, 1/3). Angles: 135°, 135°, 90°, 90°, 135°, 135°. This is an interesting hexagon because three sides have the same lengths so there are many ways to tile the hexagons together.

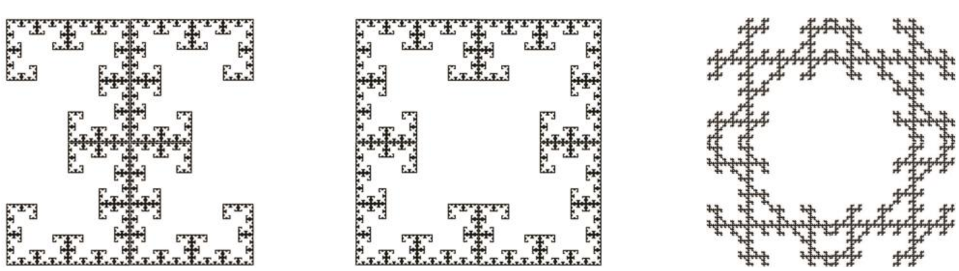
Fractals Tiled with Sierpinski Relatives

Here we present a selection fractals created by tiling together copies of one particular symmetric relative. They follow the rule that if the convex hulls of two tiles intersect, they intersect in a line segment of the same length. This is not a complete presentation, just a highlight of ones that I liked. Many of these fractals are "gasket fractals" (bounded, multiply-connected fractals) as described by Fathauer [3]. Many possess the same symmetry properties as the square: Figures 9(a) and (b); Figures 10(b) and (c); Figures 11(a), (b) and (c); Figures 12(a) and (c); Figure 13(c). The fractals in Figures 10(a) and 12(b) have four symmetries (the identity, rotation by 180°, horizontal reflection and vertical reflection). The fractals in Figures 13(a) and (b) display a chirality in that they are mirror images of each other; each of these fractals has the four rotational symmetries of the square, but no reflective symmetries. Note that the fractals are not technically self-similar because they are not formed from smaller versions of themselves.



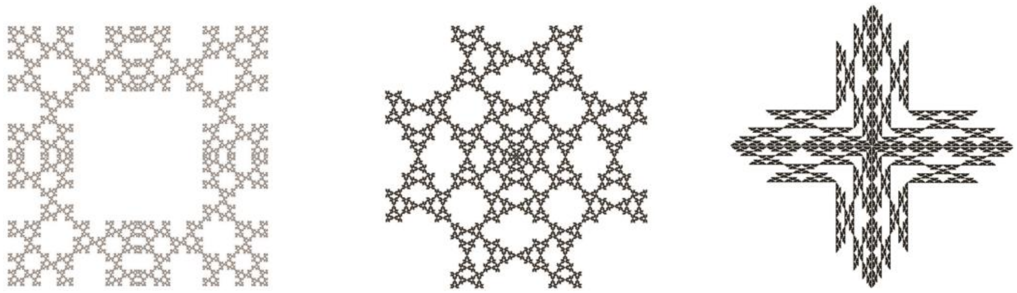
(a) (b)

Figure 9: *Fractals tiled with the Sierpinski gasket R_{aaa}.*



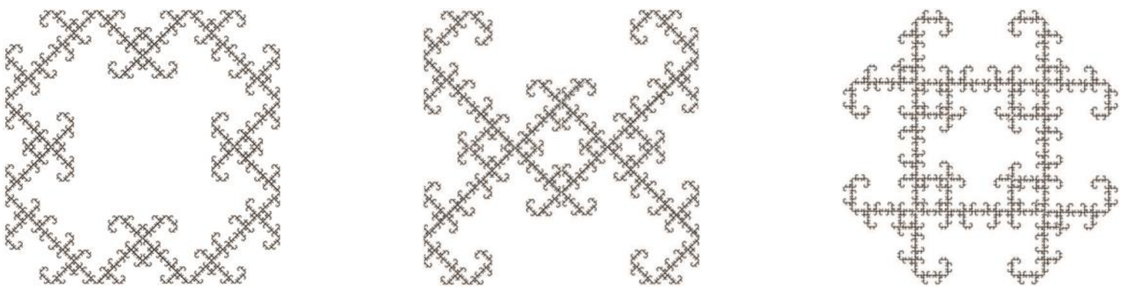
(a) (b) (c)

Figure 10: *(a) and (b) Fractals tiled with R_{abd}, (c) Fractal tiled with R_{ccc}.*



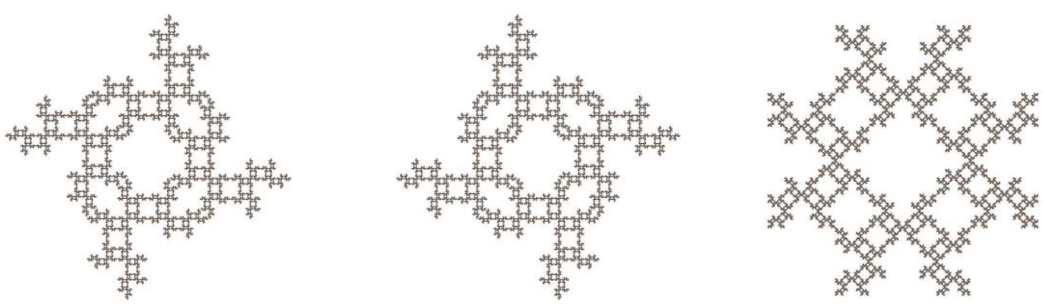
(a) (b) (c)

Figure 11: *Gasket fractals using (a) R_{acc} , (b) R_{adb}, and (c) R_{caa}.*



(a) (b) (c)

Figure 12: *Gasket fractals tiled with R_{cbd}.*



(a) (b) (c)

Figure 13: *Gasket fractals tiled with R_{cdb}.*

Frieze Patterns

The convex hulls can be tiled together to form frieze patterns. A frieze pattern is an infinite strip with a repeating pattern [1,4]. All frieze patterns possess translational symmetry. There are seven possible frieze patterns that depend on which other symmetries are present. The other possible symmetries are glide reflection (GR), vertical reflection (VR), horizontal reflection (HR) and half-turn rotation (HT). Below are a selection of examples of fractal frieze patterns formed from symmetric Sierpinski relatives. Figure 14 possesses HT symmetry. Figures 15 and 18 possess GR, VR and HT symmetries. Figure 16 possesses VR symmetry. Figures 17 and 19 possess GR, VR, HR and HT symmetries. Figure 20 possesses GR and HT symmetries.

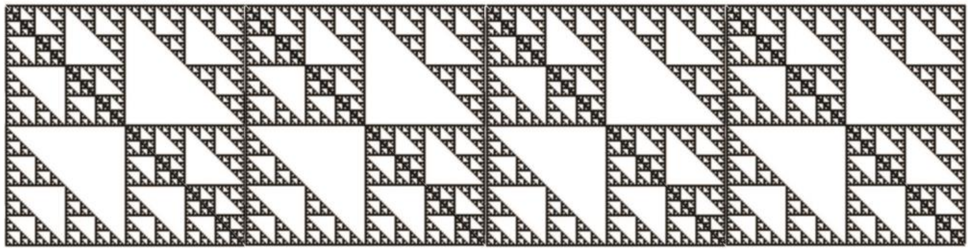


Figure 14: *Frieze pattern using* R_{aaa}.

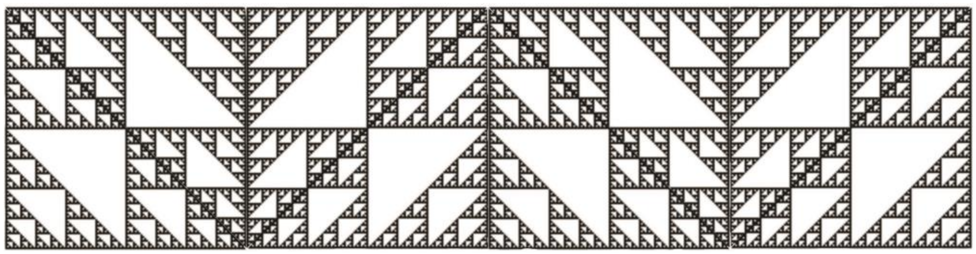


Figure 15: *Frieze pattern using* R_{aaa}.

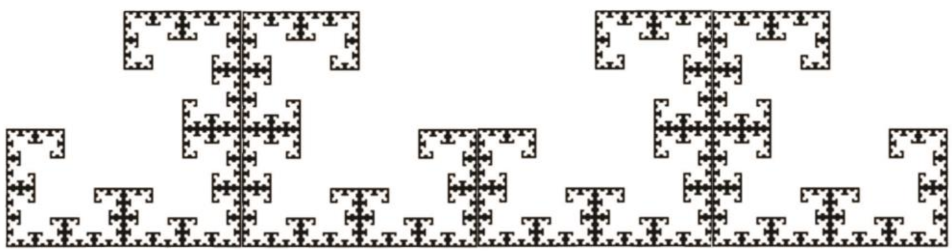


Figure 16: *Frieze pattern using* R_{abd}.

Figure 17: *Frieze pattern using* R_{adb}.

Figure 18: *Frieze pattern using* R_{caa}.

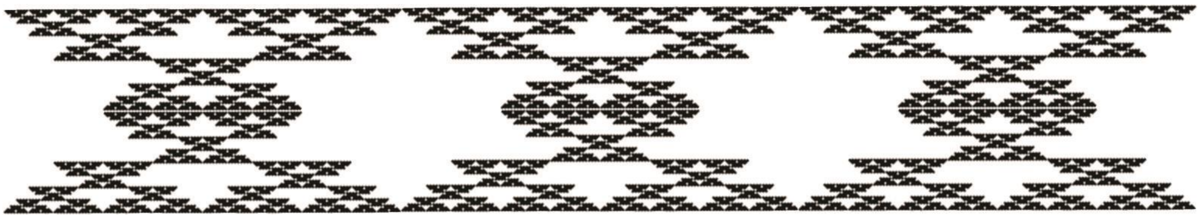


Figure 19: *Frieze pattern using* R_{caa}.

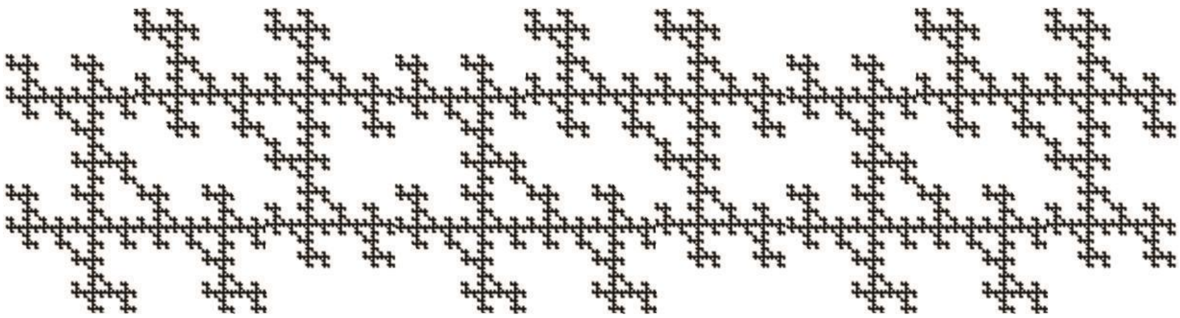


Figure 20: *Frieze pattern using* R_{ccc}.

Summary and Conclusions

This paper has presented a way to use the convex hulls of a class of fractals to generate other fractals. More work can be done with the Sierpinski relatives by looking at the non-symmetric relatives. This technique could be applied to other fractals as well.

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