Weaving Double-Layered Polyhedra

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Abstract

At Bridges 2016 I showed single sheet folding nets of some of the Platonic double layer single surface polyhedra. Having real physical models helps to understand the complex structure of these double layer single surface polyhedra. Therefore I developed more simple ways to create these models. In this paper I want to introduce weaving as a technique for creating models of double layer single surface polyhedra.

Introduction

In Figure 1 we can see how the double layered cube (Figure 1d) is developed starting with the elevated cube (Figure 1a) as is shown in Leonardo's illustration in the book Divina Proportione by Luca Pacioli [4]. In Leonardo's drawing we see an object which is made up of 6 x 4 = 24 triangles. The basic shape, the cube, can still be recognized. We can imagine that there is a cube inside, which leads to my interpretation of the object: a cube surrounded by 6 x 4 triangles as shown in Figure 1b. By choosing the connections of the faces as shown in Figure 1c, we can get the double layered cube of Figure 1d, hereby transformed in a single surface construction. I have presented this kind of double layered polyhedra in my 2008 Bridges paper "Connected Holes" [5] and at Bridges 2016 I showed some examples of paper models of double layered polyhedra.

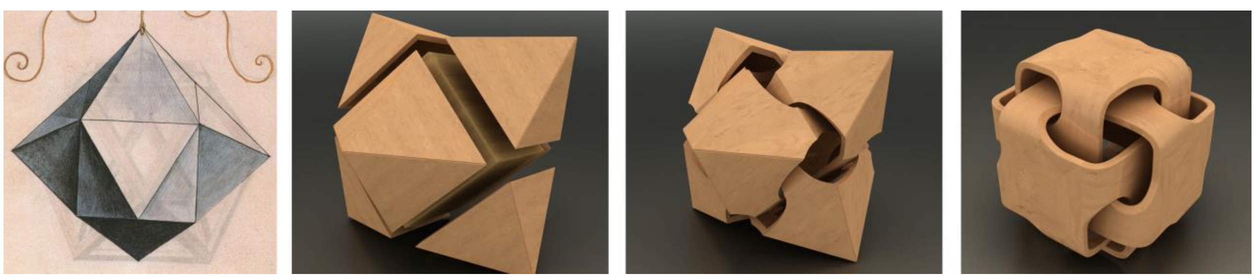

Figure 1: *(a) Leonardo's drawing. (b) All 24+6 faces. (c) New connections. (d) Double layered cube.*

Note that the resulting polyhedron is not a regular polyhedron because we don't have equal regular faces anymore. The squares will become equal when we minimize the distance between the outer and the inner faces, but in this paper we want to keep a visible distance between the layers. In his paper "Are Your Polyhedra the Same as My Polyhedra" Branko Grünbaum discusses double layered polyhedra which are regular polyhedra [3]. At Bridges 2016 I showed that it is possible to make paper models of the double layered cube from a single sheet plan (Figure 2a). Also for other double layered Platonic solids it appeared to be possible to make single sheet nets (Figure 2b,c). It turned out to be a nice challenge to find out how to fold these nets in the right way, but also a difficult task. Because of the importance of having models of these complex objects, I wanted to look for a simpler way to make them.

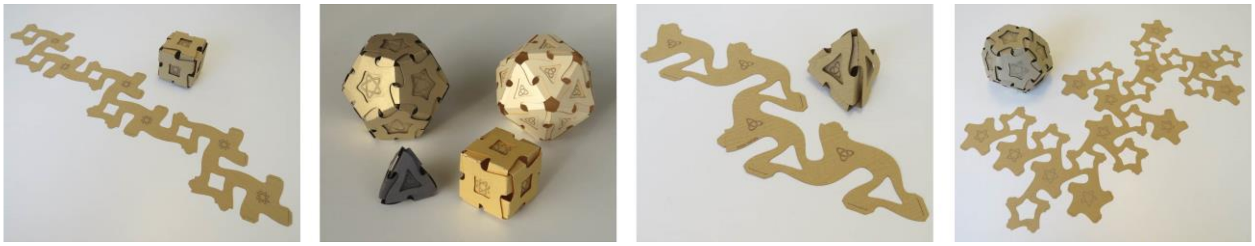

Figure 2: *(a) Double layered cube net. (b) Double layered Platonic solids. (c) Single sheet net for the double layered tetrahedron. (d) Single sheet net for the double layered dodecahedron.*

Unfolding the Double Layered Cube

Just like the normal cube we can unfold the double cube by making some cuts along the edges. The double layered cube then can be unfolded in the way showed in Figure 3.

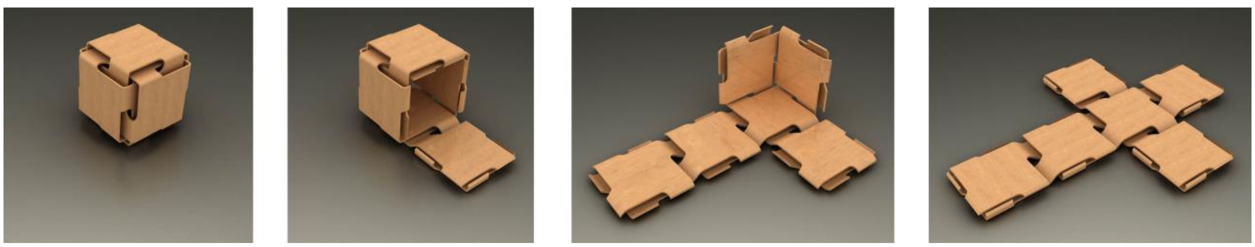

Figure 3: *Unfolding the double layered cube.*

Looking at the result of the unfolded double layered cube we can see that the net is similar to the net of a simple cube, first presented by Albrecht Dürer in his book Unterweisung der Messung [2] (Figure 4a). But the unfolding of the double layered cube consists of 6 double faces instead of 6 simple faces (Figure 4b). When we look close we see that the unfolding is an interwoven structure, and can be seen as a cut out part of the interwoven square tiling of Figure 4c.

Figure 4: *(a) Dürer's cube net. (b) Net of the double layered cube. (c) Interwoven square tiling.*

Having seen this, we can go the other way around: we can start with 2 normal cube nets, weave them together and then fold the interwoven set to a double layered cube. A few remarks: each of the individual cube nets consists of faces for the inner cube and faces for the outer cube. These faces alternate in each of the plans (In the example of Figure 5, the outer faces are the faces with the holes). When we weave the two nets together, in the end result all the outer faces have to be at one side (In the example of Figure 5, in the third picture from the left, you can see them on top). The process can be describes as follows: step 1:

put the white net on top of the grey net and, by rotating, get the grey faces with the holes on top of the white faces without the holes (as illustrated in the second picture of Figure 5. After that, we have to complete the weaving process by bringing one more white face with a hole to the top. And now we can assemble the double layer cube by folding all the double faces in the right way.

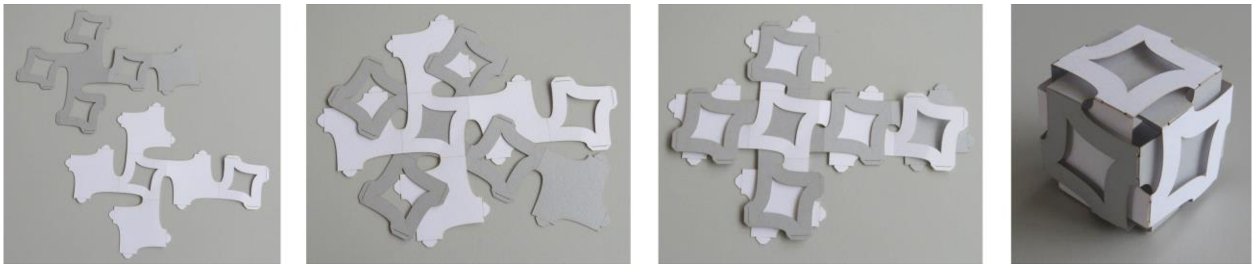

Figure 5: *Making a model of the double layered cube.*

Because with this method we make use a net of the simple cube to construct the double layered cube, we have got 11 different possible choices for our double nets (see Figure 5, first picture). When we compare the set of nets of the example in Figure 5 with the example in Figure 6 we can see that in the second example each element of the set has 3 inner faces and 3 outer faces for the double layered cube. And both elements are the same. So in this example only 1 type of element is needed. The basic cube net of this example can be seen as 1 strip of squares. It is one of the 4 cube nets without a T-junction. Weaving two copies of one of these ‘strip-nets’ is a very easy task.

Figure 6: *Alternative plans for cube and double layered cube.*

Double Layered Tetrahedron

The double layered cube is constructed by taking a part of the interwoven square tiling in the shape of a net of a simple cube. For the double layered tetrahedron we can use the same approach, but now starting with the interwoven triangular tiling (Figure 7a). There are 2 different nets for the simple tetrahedron. One of them is shown to us by Dürer (Figure 7b). In Fig. 7c and 7d you can see the parts of the interwoven tiling, needed to fold the double layer tetrahedron.

Figure 7: *(a) Interwoven triangular tiling. (b) Dürer’s tetrahedron net. (c),(d) double tetrahedron nets.*

The steps of the weaving process based on net presented by Dürer can be seen in the first three pictures of Figure 8. In the fourth picture of Figure 8the final result is shown.

Figure 8: *Weaving and folding the double layered tetrahedron based on Dürer's net.*

When we use Dürer's net, both elements will be equal. In the other case both elements will be different, as can be seen in Figure 9. The weaving process in this case is based on rotation.

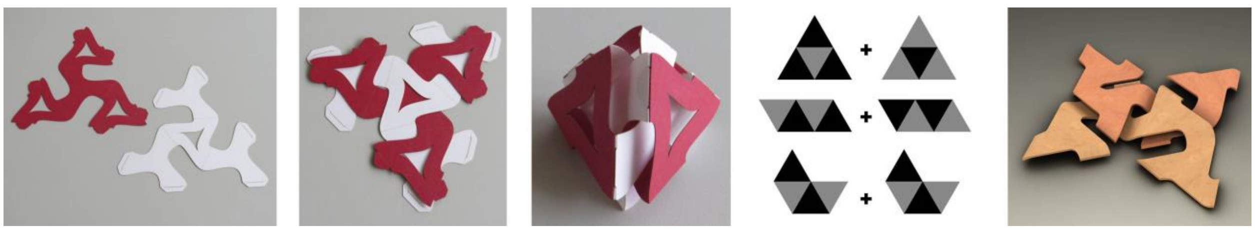

Figure 9: *Alternative plans for the double tetrahedron.*

There are more possibilities to create sets of nets with which we can make a double layered tetrahedron. One of them is shown in the most right picture of Figure 9. In this paper however we will limit ourselves to designs based on nets of the simple polyhedra.

Double Layered Cuboctahedron

Dürer also showed nets of some of the Archimedean solids. In the first picture of Figure 10 we see Dürer's drawing of the plan of the cuboctahedron. The cuboctahedron is built with squares and triangles. And indeed, we also have an Archimedean tiling with squares and triangles. It is interesting to see that Durer's net is a fragment of this tiling. So it looks like we can use the same approach as we did for the double layered cube or the double layered tetrahedron to get the double layered cuboctahedron.

Figure 10: *Plan of the cuboctahedron.*

But when we double the tiling, as we did with the square tiling and the triangular tiling, the result now isn't a weaving of two layers. Everything seems to be connected. And after folding the double cuboctahedron we don't get one connected object, but a compound of two separate structures. This has to do with the odd or even number of tiles coming together in a vertex, which is explained in my paper

'Connected Holes', Bridges 2008 [5]. The same effect will show up when we double the octahedron, also then the end result will be a compound.

Double Layered Dodecahedron

It's not possible to make a regular tiling with regular pentagons. So for the net of the dodecahedron we cannot take a part of a tiling. But we can directly double any net of a simple dodecahedron to create a double layered dodecahedron. The net Dürer has chosen is quite familiar and has nice symmetries. For the weaving process we can look for other properties. Looking at the graph of Dürer's choice, we can say that this is the choice with the highest vertex values. The graph with the lowest vertex values is the representation of the strip net, used for the design of the fourth picture in Figure 11. Making this one is like really weaving two strips together.

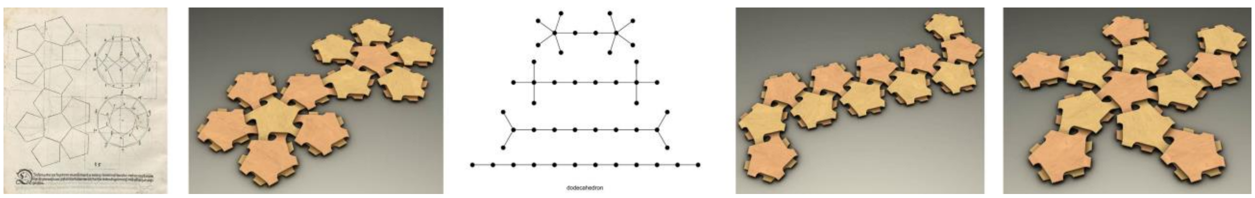

Figure 11: *Net of the dodecahedron.*

In the other examples (Figure 11, fifth picture and Figure 12) two star-shape solution is shown. For the last one, both parts of the construction are equal.

Figure 12: *Model of the double layered dodecahedron.*

Double Layered Icosahedron

Also for the double layered icosahedron there are many different basic nets we can use. In total 43380. In Figure 13a we see Dürer's choice which is used for the design of the model of the double layered icosahedron in Figure 13d. In Figure 13b en 13c the parts and the weaving of the parts are shown.

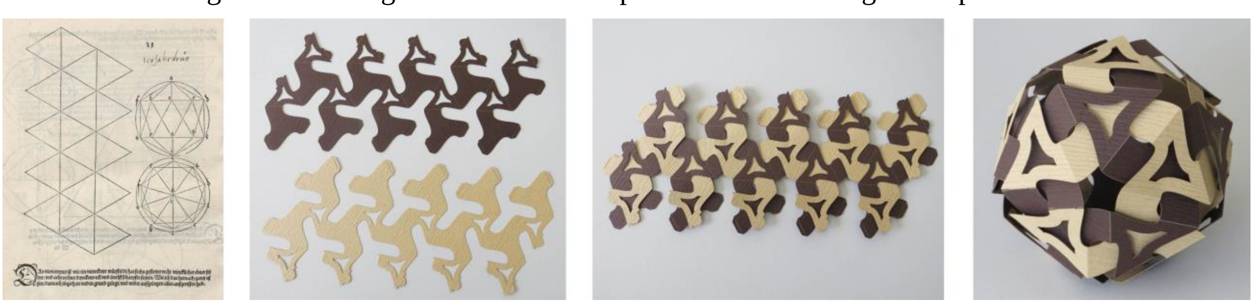

Figure 13: *(a) Net of the icosahedron. (b),(c),(d) Parts, weaving and final double layered icosahedron.*

In Figure 14 two more solutions for the double layered icosahedron are show. The second example is again based on a strip folding net. Also the third example in Figure 14, the double layered truncated octahedron is based on a strip folding net.

Figure 14: *Cross net and single strip nets for double layered polyhedra.*

We now have developed a general method to construct nets for double layered polyhedra. This weaving concept will work for all polyhedra which have at least one vertex in which an odd number of faces come together. As a final example the construction of the double layered truncated octahedron is shown.

Elevation – Combinations of Polyhedra

The double layered polyhedra which we studied so far are in fact a simplification of my interpretation of the Pacioli's elevated polyhedra. Let's go back to the original elevated polyhedra and see how the concept of weaving can be applied to create models of the double layered polyhedra derived from Pacioli's elevated cube.

Figure 15: *Model based on Leonardo's drawing of the elevated cube. Fold – Weave – Fold.*

Basically we can take the same steps as in the construction of the double layered cube in Figure 6. There is however one step we have to take before we start the weaving: first we have to fold the pyramids, the elevations, in both parts of the model. This is shown in the second third and fourth picture of Figure 15. So first fold, then weave and then fold again. A three step process. This process also opens another possibility of making variations of the elevated polyhedra: we can vary the height of the elevation in such a way that the outer skin of the construction gets the shape of another polyhedron. In the outer skin of the model in the final picture of Figure 15 we can recognize the rhombic dodecahedron.

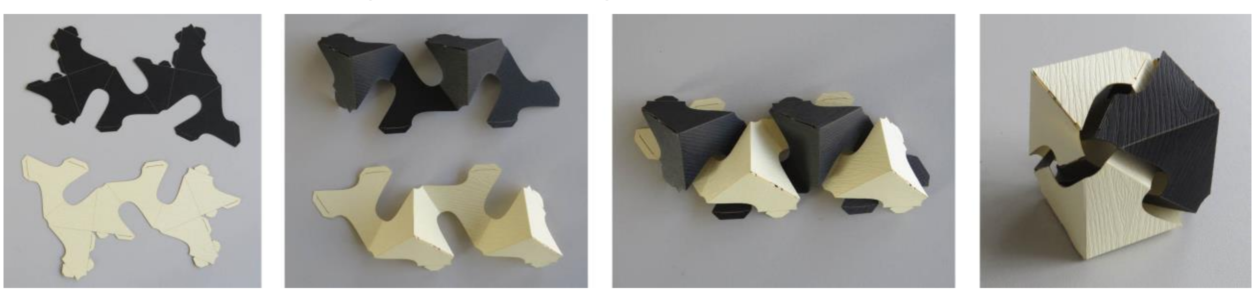

Figure 16: *Model based on Leonardo's drawing of the elevated tetrahedron. Fold – Weave – Fold.*

The basic inspiration for the model in Figure 16 was Leonardo's elevated tetrahedron. In this model the height of the elevation is changed in such a way that the outer skin becomes a cube. Indeed, we know we can construct a tetrahedron inside a cube by using the face diagonals of the faces of the cube. With this model this is illustrated in a nice way.

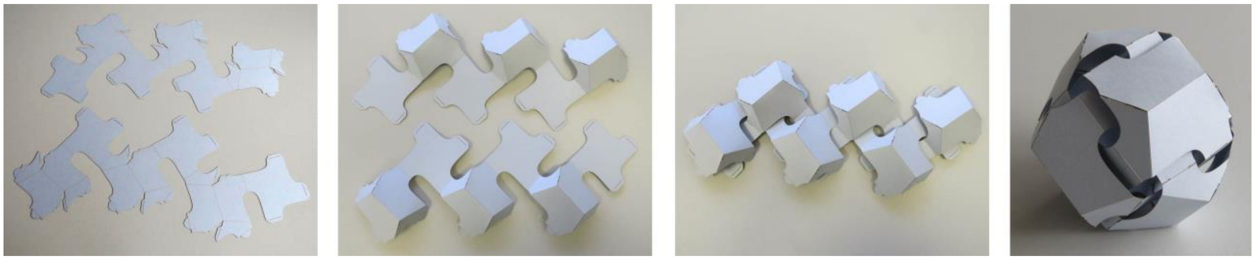

Figure 17: *Variation of the double layered cube: combination of cube and dodecahedron.*

We also know that we can construct a cube inside the dodecahedron by using face diagonals of the faces of the dodecahedron. And also this can be illustrated with a model of a double layered polyhedron. The parts, the folding of the parts, the weaving and the final model are shown in Figure 17. So again the three step process: folding, weaving, folding.

Figure 18: *Model based on Leonardo's drawing of the elevated icosahedron. Fold – Weave – Fold.*

Another nice and interesting property of this three step process is that the relation between the flat parts and the final double layered polyhedra is not obvious and sometimes unexpected. An example based on the elevated icosahedron is shown in Figure 18. Only after the first folding step we can recognize the net of a icosahedron.

Strips and Rings

There is one other association that appeared to be fruitful in the design process of weaving models of double layered polyhedra. The structure of the double layered cube shown in the first picture of Figure 19 has much in common with the Borromean ring structure, the three interwoven rings in the second picture in Figure 19. We can divide the 2 x 6 faces of the double layered cube into 3 strips of 4 faces. In the final model we still can recognize the three strips, now as three interwoven rings. The total object however is the same as the double layered cubes of Figure 5 or Figure 6.

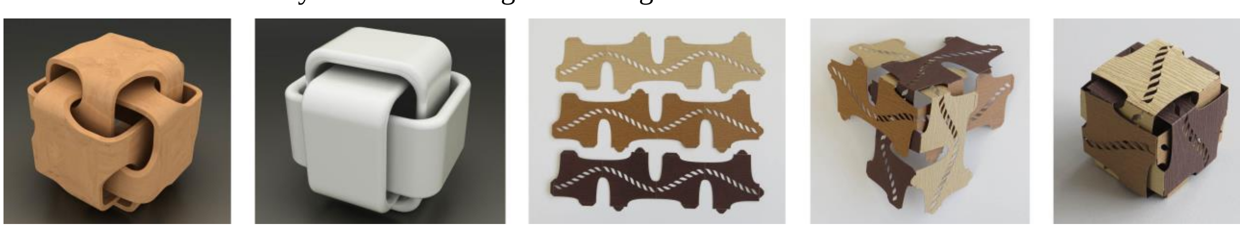

Figure 19: *(a) Double layered cube. (b) Three interwoven rings. (c),(d),(e) Weaving with strips.*

Also the combination polyhedron cube/dodecahedron of Figure 17 can be made as a three ring construction. This is shown in the first two pictures of Figure 20.

Figure 20: *Strip weaving: cube/dodecahedron (3 strips) and double layered dodecahedron (4 strips).*

The real double layered dodecahedron consists of 2 x 12 = 24 faces. We can divide this in 4 rings of 6 faces and then construct the 4 rings double layered dodecahedron as shown in the last picture of Figure 20.In the same way we can make a model of the double layered rhombic triacontrahedron with 6 strips and a model of the double layered dual of the rhombic cuboctahedron. The final model shown here is the double layered dual of the rhombic icosidodecahedron. It is constructed out of 10 equal strips, interwoven through each other. Note that the strips have to be connected to each other after weaving.

Figure 21: *Strip weaving: double layered dual of the rhombic icosidodecahedron (10 strips).*

Conclusion

The techniques developed here to make models of the double layered polyhedra may have some similarities with plaiting as described by Cundy and Rollett in Mathematical Models [1]. This is also a weaving technique, but it does not result in models of the double layered single surface polyhedra. I think the weaving concept as it is developed here is a very nice way to construct models of the double layered single surface polyhedra. It makes it easier to make the models of these complex structures. Although the models of the double layered polyhedra are in fact single surface structures, in most of the examples I have used more than one color to explain to process.

References

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