

Torus and Klein Bottle Tessellations with a Single Tile of Pied de Poule (Houndstooth)

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Abstract

We design a 3D surface made by continuous deformation applied to a single tile. The contour edges are aligned according to the network topology of a Pied-de-poule tessellation. In a classical tessellation, each edge is aligned with a matching edge of a neighbouring tile, but here the single tile acts as a neighbouring tile too. The continuous deformation mapping the Pied-de-poule tile to the 3D surface preserves the staircase nature of the contour edges of the tile. It is a diffeomorphism. The 3D surface thus appears as a torus with gaps where the sides of the tile meet. Next we present another surface, also a single Pied-de-poule tile, but with different tessellation type, a Klein bottle. Both surfaces are 3D printed as innovative art works, connecting topological manifolds and the famous fashion pattern.

Introduction

Pied-de-poule (houndstooth) denotes a family of fashion patterns, see the Bridges 2012 paper [3] and Abdalla Ahmed's work on weaving design [1] for more background information. In this project, we study tessellations obtained by continuous deformation of a Pied-de-poule's basic tile. In particular, we construct tessellations with a single copy of one basic tile. The network topology of the tessellation was analysed in [3]. A typical piece of Pied-de-poule fabric can be seen in Figure 1 (left). In the system of Heesch and Kienzle [5] it has Grundtyp TTTTTT (six sides, pairwise Translated).

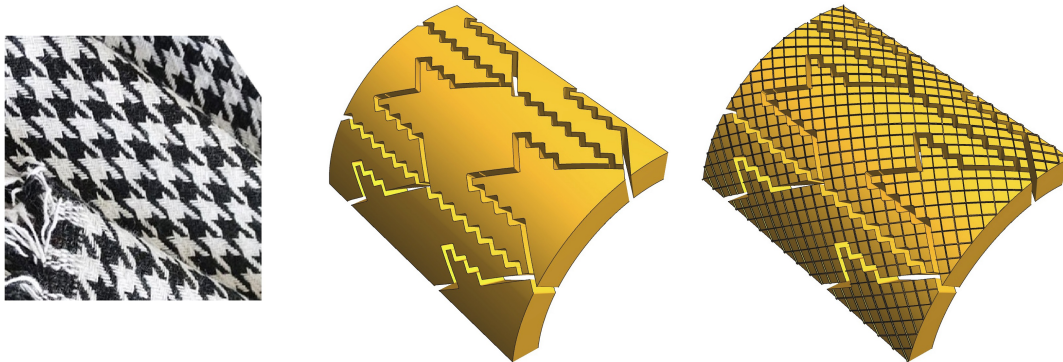


Figure 1: Pied-de-poule fabric (left), abstract tiles on a cylinder surface (center) and adding a grid (right).

In Figure 1 a basic tile is tessellated and curved along a cylindrical surface. If the surface is deformed further, it will form a full cylinder. The edges of the tile will match, and we have a folded tessellation. It can be extended along the longitudinal axis by repetition. Next, the cylindrical tube can be continuously deformed into a torus, connecting open ends, and the tiles will still tessellate. There are different approaches of aligning the tiles with respect to the cylinder, two of which are in Figure 2. In this project we choose the second approach (Figure 2 right).

What is the minimal number of tiles needed, demanding that it is still a tessellation? It turns out that *one* tile is sufficient. This is well-known for a rectangle, such as a Pacman playing field, with tessellation Grundtyp TTTT, but for our TTTTTT tessellation it works as well. The first part of this article is about

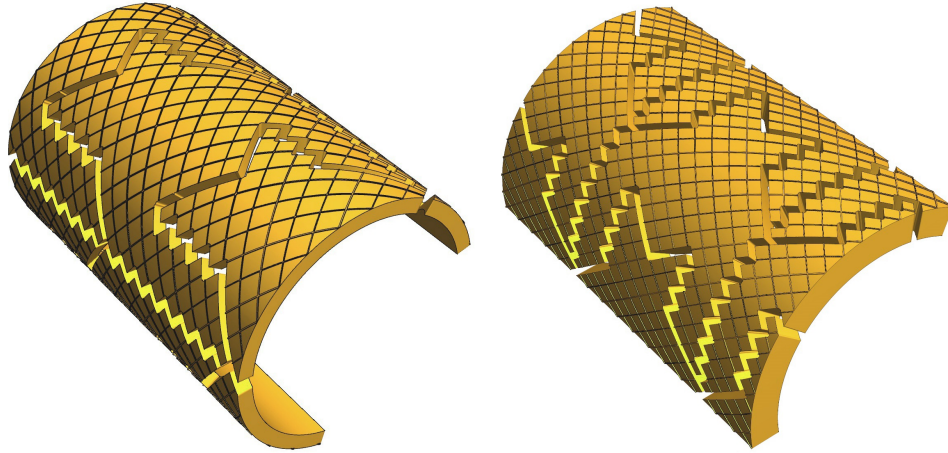


Figure 2: Two ways of tiling basic Pied-de-poule tiles on a cylinder surface.

making such a one-tile tessellation. Figure 3 shows eight snapshots from an animation folding the tile into a torus. The torus is not fully closed but has a gap which is left open to indicate where the sides of the tile meet. The gap consists of 45 segments which follow straight lines and staircases, precisely as the contour of the Pied-de-poule tile. The goal is to create a 3D printed version of the resulting torus. Before we can really make such a torus, we have to work through several mathematical technicalities, also by way of preparation for the article's more difficult second part: tiling on a Klein bottle.

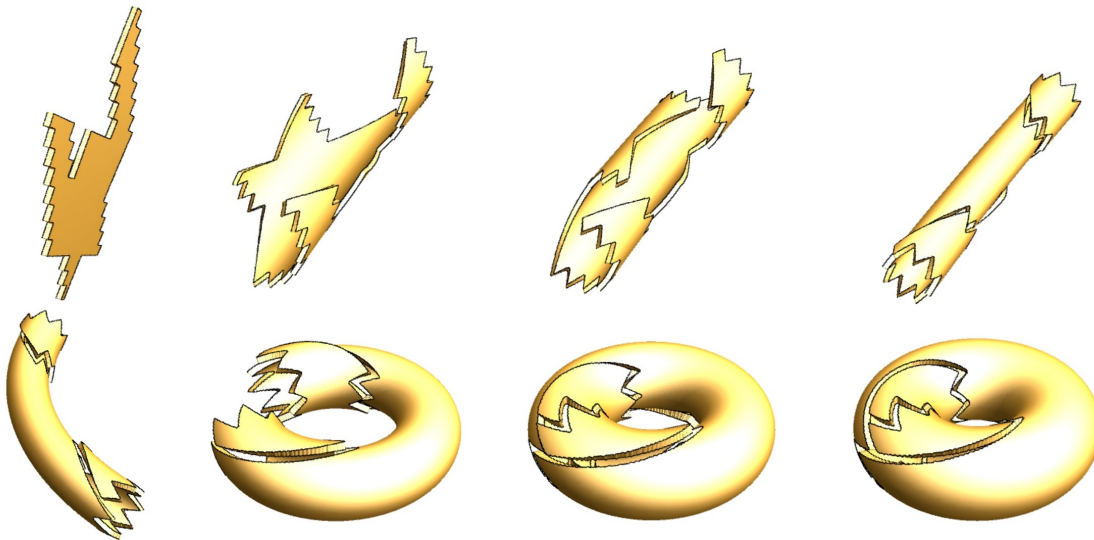


Figure 3: Folding a Pied-de-poule tile into a tube (upper row) and folding the tube into a torus (lower row).

Implementing the Tile on a Torus

First, we need equations for the torus, then equations for the positions of the segments of the gap. A practical parametric description of the x, y, z coordinates of a torus is $x(u, v) = (\cos u)(2 + \cos v)$, $y(u, v) = (\sin u)(2 + \cos v)$, and $z(u, v) = \sin v$. Mathematica's `ParametricPlot3D` can render this parametric description as a 3D surface. The parameters are u (longitude) and v (azimuth) and both range from 0 to 2π . So there is a rectangular (u, v) coordinate system on the torus surface. Next, introduce another coordinate grid, which is at 45° relative to the u and v , see Figure 1 (right) and Figure 2.

Taking $p = 8(u + v)/(2\pi)$ and $q = 8(u - v)/(2\pi)$, the desired coordinate system appears. The (p, q) coordinate system is convenient as the staircase steps and straight line segments of the Pied-de-poule's contour follow either lines of constant p or lines of constant q . We use the coordinate system to define the gaps (left open during 3D printing). Now $0 \leq p < 16$ and $-8 \leq q < 8$, although only about half of the 256 integer (p, q) combinations are valid (enough for one tile). A typical gap segment is defined as $p = 13$ and $-3 \leq q \leq -2$ and then the next, going around a corner is $12 \leq p \leq 13$ and $q = -2$, and so on (45 steps). Combining all these one-step gap segment definitions gives a predicate, say gapP . Care is needed when crossing a "date line", which occurs for $u = 2\pi$ ($u = 0$), and $v = 2\pi$ ($v = 0$). The set T is defined to contain all points $(x(u, v), y(u, v), z(u, v))$ on the torus, except for the points for which $\text{gapP}(u, v)$ holds.

For practical 3D printing, the points having a distance $\leq \varepsilon$ to any of these gap segments should be excluded (and the gaps extended to avoid rounded corners). This gives another predicate, gapP' say. We take $\varepsilon = 0.025$ in $(u + v, u - v)$ space. Mathematica option `RegionFunction` $\rightarrow \neg \text{gapP}'$ executes that.

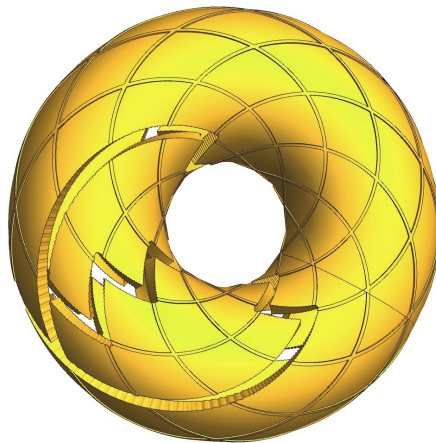


Figure 4: Using a gap to define a basic Pied-de-poule tile on a torus.

After having presented this construction, it is time to be more precise about our claim. In what sense is the deformed tile equivalent to the original? Let's call the original Pied-de-poule tile P , and the tile deformed into the torus-with-gap T . Certainly, P and T are homeomorphic. Two objects are *homeomorphic* if one can be deformed into the other by a continuous mapping that has a continuous inverse (like stretching rubber). It is enough to check that the presence of the gaps has changed the torus into a tube and also opened-up the resulting tube to become one connected surface. Regretfully, the claim that the surfaces are homeomorphic, is too weak to be interesting. Even a unit disc is homeomorphic to a Pied-de-Poule, though the two do not resemble one another. What makes T still Pied-de-poule-like? The answer is that P and T are *diffeomorphic*. A diffeomorphism is a map between manifolds which is differentiable and has a differentiable inverse [9]. We write *smooth* to mean diffeomorphic, although most authors adopt a stronger notion of smoothness (check higher order derivatives too). Intuitively, the homeomorphism properties guarantee that no new holes are created and no distant points are merged. The diffeomorphism guarantees that straight lines are mapped to smooth curves (no sharp bends) and that corners are mapped to corners, although angles may differ. A diffeomorphism from P to T would preserve the edges and the staircase nature of the contour.

Does such diffeomorphism exist? Yes, since we can first consider P as a contour in Cartesian (p, q) coordinates, and then identify these p and q with the coordinates used for defining gapP . The (p, q) coordinates are related to the (u, v) coordinates via linear relations, for example $p = 8(u + v)/(2\pi)$. The latter coordinates are related to the (x, y, z) of three-dimensional Euclidean space by differentiable trigonometric functions such as $x = (\cos u)(2 + \cos v)$. Finally, it can be checked that T itself is smooth, which follows from the smoothness of the torus at the date line $u = 2\pi$, identified with the line $u = 0$. The periodicity and differentiability of sine and cosine guarantee that T is differentiable, and so is our mapping.

Implementing the Tile on a Klein Bottle

We found another remarkable construction to fold the tile. It has Heesch-Kienzle Grundtyp $TG_1G_2TG_2G_1$, which means a pair of Translated edges and two pairs Glide reflected. The long staircase is aligned with itself reversely. The tile fits on a Klein bottle, the famous topological surface invented in 1882 by Felix Klein [6]. It is not trivial that this one-tile tessellation is feasible. Traditional topology texts introduce a Klein bottle as a square, e.g. Bredon [2] p.43: “The Klein bottle is a square with opposite vertical edges identified in the same direction and opposite horizontal edges identified in the opposite direction.” But our basic tile has six edges, not four, see Figure 5. First we model a smooth Klein bottle, then we find the coordinates of the gap.

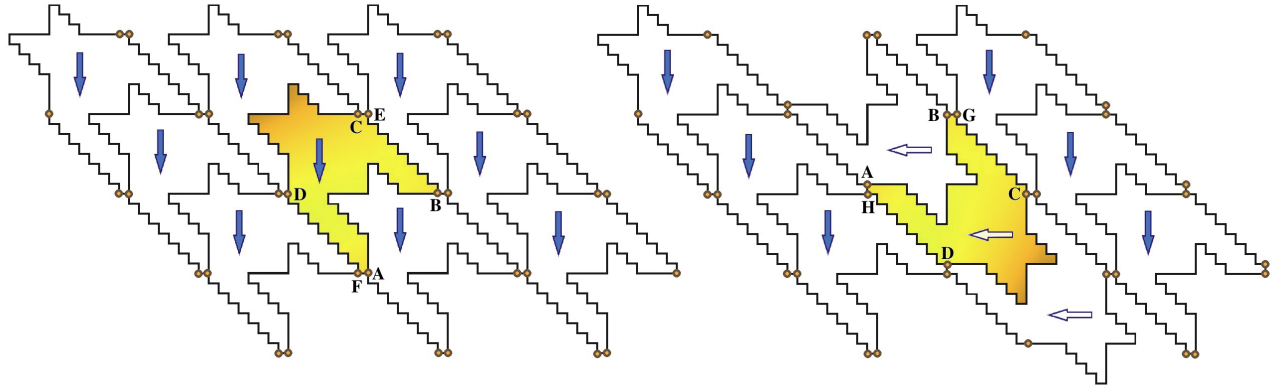


Figure 5: Classical $TTTTTT$ tessellation with translations $AB \rightarrow DC$, $AF \rightarrow EC$, $BE \rightarrow FD$ (left) and new $TG_1G_2TG_2G_1$ tessellation with translation $AB \rightarrow DC$ and glide reflections $HD \rightarrow CG$ and $AH \rightarrow GB$ (right).

Heikki Ruskeepää [7] (p. 151) gives a parametric description for the x, y, z of a Klein bottle using parameters u, v from 0 to 2π . Let $a = (6 \cos u)(1 + \sin u)$, $b = 16 \sin u$, and $c = 4(1 - (\cos u)/2)$, then

$$x(u, v) = \begin{cases} a + c \cos(v + \pi) & \text{if } \pi < u \leq 2\pi \\ a + c \cos u \cos v & \text{otherwise} \end{cases}$$

Next $y(u, v) = b$ if $\pi < u \leq 2\pi$ and $y(u, v) = b + c \sin u \cos v$, otherwise. Finally $z(u, v) = c \sin v$. Again, Mathematica’s `ParametricPlot3D` renders this. Think of u as the longitudinal axis of a cylindrical tube, v the azimuth. This tube then is bent to enter itself, folded outward and finally is self-connected again. Aiming at a diffeomorphism mapping our tile to a “Klein bottle-with-gaps”, we noticed a problem: Heikki Ruskeepää’s bottle equations violate smoothness at $u = \pi$ and (somewhat less) near the date line $u = 2\pi$ ($u = 0$), see Figure 6. The jumps in the derivative can be seen in Figure 7 (right). We shall repair them.

Can we make continuous functions into smooth functions? Over a finite interval create its Fourier series to as many terms as you are happy with (answer by ‘Paul’ on math.stackexchange.com question 1780870). Figure 7 shows that the smoothness violation is in the $y(u, v)$ component of the parametric description, introduced by the if-otherwise. To implement Paul’s answer $y(u, v)$ is turned into a periodic function, period 4π , and then symbolically seven Fourier coefficients are calculated, $a_0 = \frac{1}{2\pi} \int_0^{4\pi} y(u, v) du$, $a_1 = \frac{1}{2\pi} \int_0^{4\pi} y(u, v) \cos t dt$ and similarly a_2, a_3, b_1, b_2 , and b_3 . The seven-term reconstruction is $y_s(u, v) = \frac{1}{2}a_0 + \sum_{n=1}^3 a_n \cos \frac{1}{2}nx + \sum_{n=1}^3 b_n \sin \frac{1}{2}nx$. The terms a_0 and a_2 happen to be zero. The complete Fourier series would use $\sum_{n=1}^{\infty}$. In electrical-engineering language, we implemented a low-pass filter. Thus a smooth parameterised y coordinate formula is found (included here for readers making their own smooth bottles):

$$y_s(u, v) = \frac{24 \cos v}{5\pi} \cos \frac{u}{2} - \frac{152 \cos v}{35\pi} \cos \frac{3u}{2} + \frac{88 \cos v}{15\pi} \sin \frac{u}{2} + 16 \sin \frac{2u}{2} + \frac{72 \cos v}{35\pi} \sin \frac{3u}{2}$$

In Figure 8 the improvement can be seen. The date line $u = 2\pi$ ($u = 0$) is in the middle of the thin part of the tube (the “handle”). It can be checked that the smoothness of $x(u, v)$ and $z(u, v)$ are okay. The parametric

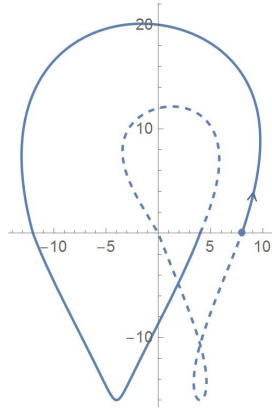


Figure 6: Parametric plot of the Klein bottle coordinates $x(u, v)$ and $y(u, v)$ for u ranging from 0 to 2π at fixed $v = 0$. For $u > 2\pi$ the natural continuation of $x(u, v)$ is found as $x(u - 2\pi, \pi - v)$ and similarly the continuation of $y(u, v)$ is $y(u - 2\pi, \pi - v)$. These continuations are dashed. After u has been running from 0 to 2π , the point (x, y) has not returned to its begin point, but now is at the opposite side of the tube.

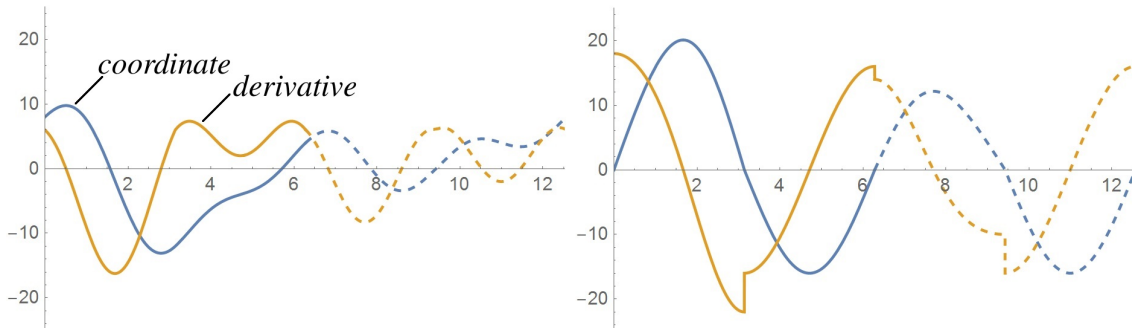


Figure 7: Parametric descriptions of the Klein bottle coordinates $x(u, v)$ and $y(u, v)$ as functions of u for fixed $v = 0$. In both left and right plots, the blue line is the coordinate itself (and its natural continuation) whereas the yellow line is its first derivative (with respect to u). Shown are $x(u, v)$, $\partial x(u, v)/\partial u$ (left) and $y(u, v)$, $\partial y(u, v)/\partial u$ (right).

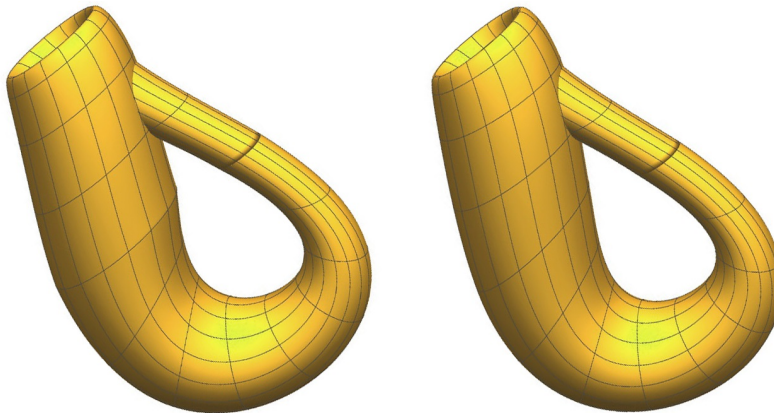


Figure 8: Klein bottle, original (left) and smoothed (right).

description is again combined with the coordinate system $p = 8(u + v)/(2\pi)$ and $q = 8(u - v)/(2\pi)$, and the gaps are implemented as before. Yet, the coordinates of the gap steps are different because (1) this is a different tessellation, and (2) the relative positioning of the date lines is different (new predicate gapQ). The gap steps are shown in Figure 9.

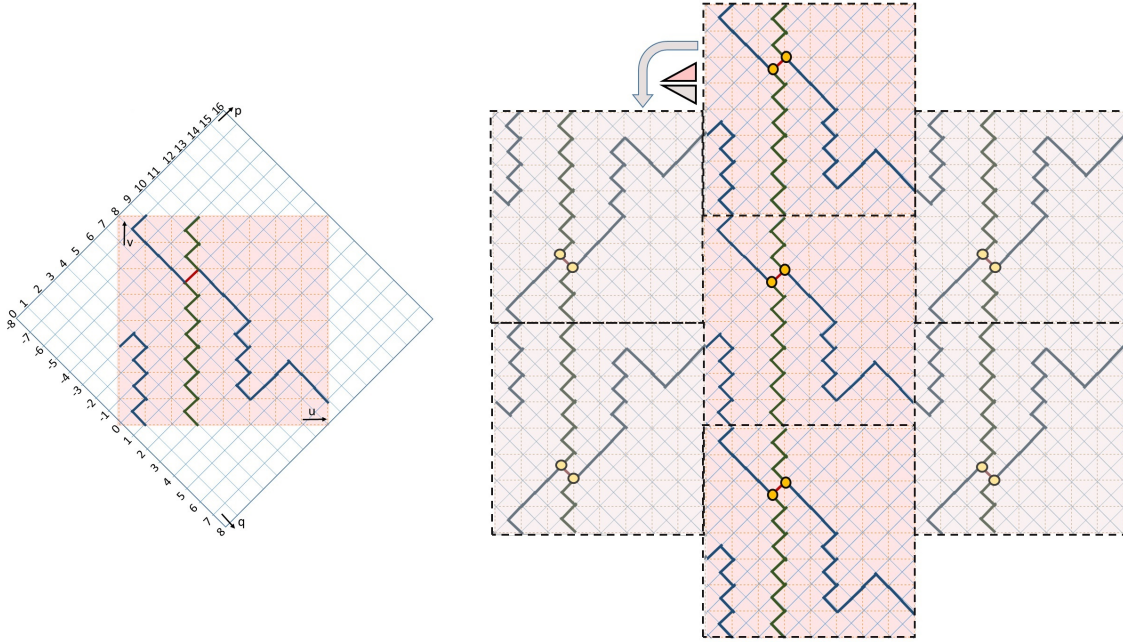


Figure 9: Positions of the segments of the gaps in the (p, q) and (u, v) coordinate systems. The central square is defined by $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$ (left). Aligning this central square with six copies, four of which are vertically flipped, the rightmost Pied-de-poule tessellation of Figure 5 becomes visible (right). Note that the line $v = 2\pi$ is identified with the line $v = 0$ in the most straightforward manner. But the line $u = 2\pi$ is identified with $u = 0$ in a reverse manner and such that the point $(u, v) = (2\pi, \pi)$ is identified with $(0, 0)$ and $(0, 2\pi)$, consistent with the natural continuations presented in Figure 6.

Let the function `klein` map each coordinate pair (u, v) to its triple of (x, y_s, z) coordinates, so we render the bottle invoking `ParametricPlot3D` for `klein(u, v)` and ranges $\{u, 0, 2\pi\}$, $\{v, 0, 2\pi\}$ with option `RegionFunction` $\rightarrow \neg \text{gapQ}'$. We found that the position of the T line (the T line in Heesch-Kienzle Grundtyp $TG_1G_2TG_2G_1$) cannot be chosen arbitrarily, some care is needed to guarantee that there is no jump at the date line $u = 2\pi$ ($u = 0$), if we want to avoid the alternative of twisting the bottle.

The Klein bottle-with-gaps, K say, is diffeomorphic to the original tile T , except that the tube self-intersects (in four dimensions this is no problem). An ant walking the surface along the gap, can execute the same sequence of `forward`, `left` and `right` actions, as a Logo turtle writing a Pied-de-poule contour (as formalized in [3]). Unlike the classical Klein bottle, the Klein bottle-with-gap is orientable.

The morphing has stretched the tile quite a bit and it is hard to recognise the original. To make better visible what happened, we decided to show the grid of (p, q) coordinates. During 3D printing, this grid is to be rendered on top of K . As we cannot vary the thickness of the surface, we work with an extra surface in grid form. For the torus that was easy (add an outer grid torus with enlarged tube radius). For the Klein bottle, it does not make sense to add a grid tube with enlarged radius, but, following a suggestion of Gianluca Gorni in [community.wolfram.com/groups topic 602050](https://community.wolfram.com/groups/topic/602050), a copy of the surface is added (twice), displaced in the orthogonal direction. The direction is found by normalizing $\partial K(u, v)/\partial u \times \partial K(u, v)/\partial v$ (an impressive formula with over 200 `cos` and `sin` terms).

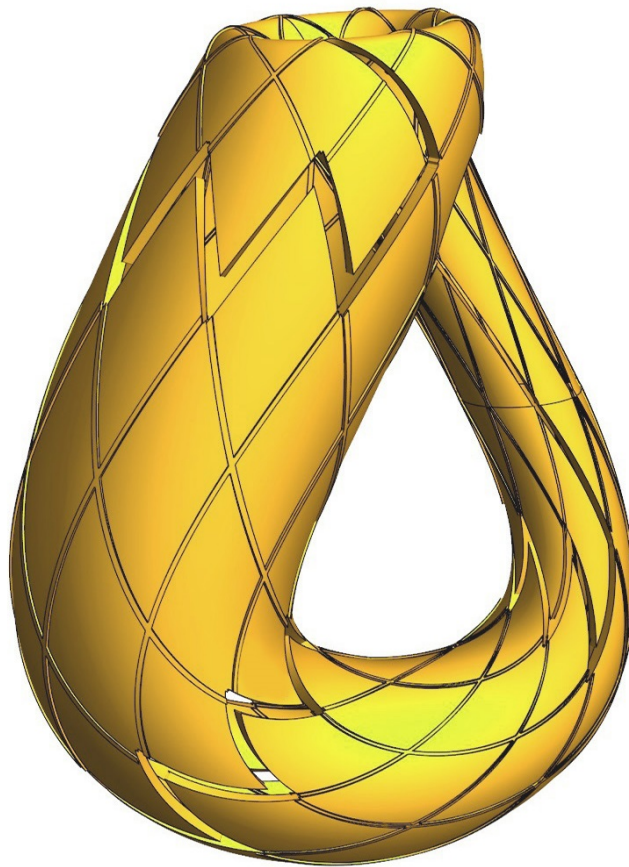


Figure 10: Mathematica rendering of the Klein bottle constructed as a morphed tile.

Concluding Remarks

Mathematica generated STL files, which were manually fine-tuned and printed by the Objet Connex 350 of TU/e ID. The grid is a thin layer of dark grey plastic on top of the hollow white body. The torus is $125 \times 125 \times 45$ mm, 4mm thick and the Klein bottle is $125 \times 197 \times 68$ mm, see Figure 11. Both are to be shown at Bridges 2018. To emphasise that the bottle-with-gaps is connected, the “outer tube” has a hole for the “inner tube”. It is fun to check that each of the objects is one P -like tile indeed.

A flat version of the one-tile $TG_1G_2TG_2G_1$ tessellation was implemented as a playing area for Conway’s Game of Life. This installation, called B3/S23 Descending a Staircase No. 2, was presented at Aplimat 2018 in Bratislava [4]. Given the great variety of possible tessellations and the unexpected variety of possible Klein bottles as described by Carlo Séquin [8], we expect that there are many more creative and artful possibilities to be explored with single-tile Klein bottles.

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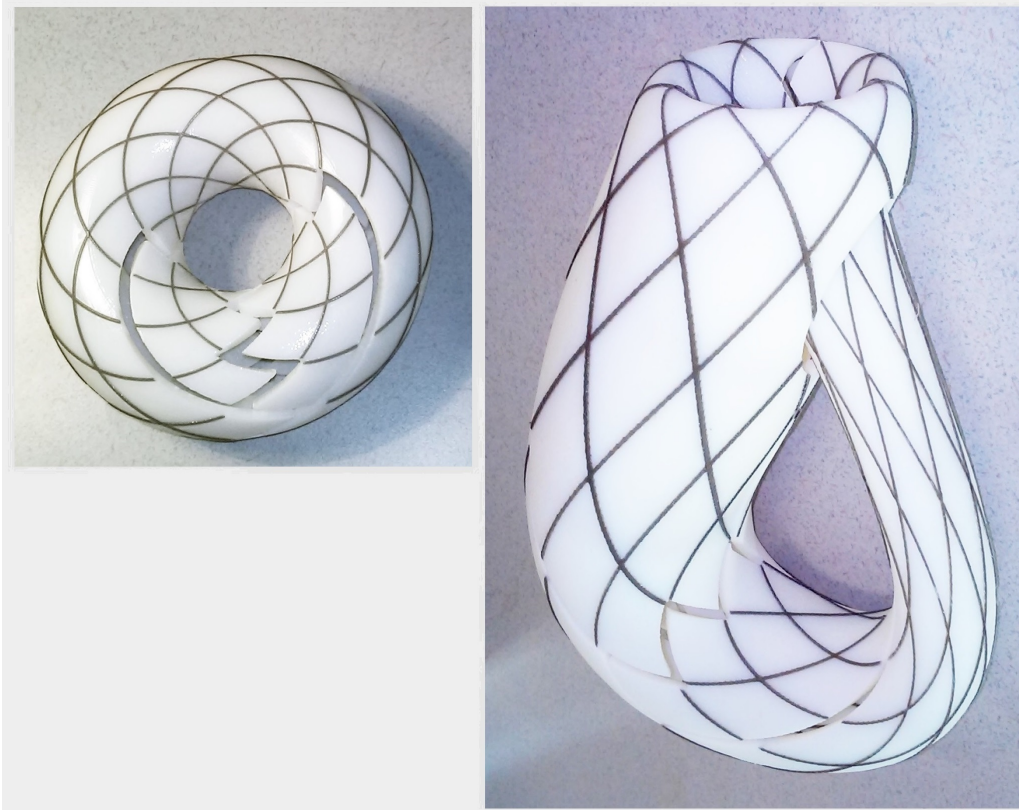


Figure 11: 3D printed torus and Klein bottle.

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