The Art and Mathematics of Self-Interlocking *SL* Blocks

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Abstract

SL block is an octocube that may interlock with other *SL* blocks to form infinite variations of stable structures. The property of self-interlocking makes *SL* block expressive to explore the beauty of symmetry, which has been regarded as an essence of art and mathematics by many. This paper describes a mathematical representation that maps polynomial expressions to compositions of *SL* blocks. The use of polynomials, functions and hierarchical definitions simplifies the creation, communication and manipulation of complex structures by making abstractions over symmetrical parts and relationships. The discovery of *SL* block and its mathematical representation lead the way towards the development of an expressive language of forms and structures which is at the same time, rich and compact, free and disciplined.

Introduction

The design of interlocking structures is interesting and useful in various fields of study and applications. To puzzle makers, the objective might be to create mind-teasing configurations that are difficult to disassemble and assemble [4][6]. Engineers and architects look for interlocking joints that can be easily assembled and yet require no nail, mortar or any other adhesive material [5][7]. For material scientists, topological interlocking opens up a broad avenue to the discovery of new and enchanted materials by configurations made up with small and symmetrical units [1][2]. In this study, self-interlocking is used to describe structures that are made up with identical parts locked with each other topologically, or less preferably, by friction. The objective is to uncover the beauty and richness of well integrated forms that can be derived systematically from one single element.

SL Block, Engagement and Strand

First proposed by Shih [3], *SL* blocks were used to create self-interlocking structures called *SL* strands. An *SL* block is an octocube consisting of an *S*-shaped and an *L*-shaped tetracubes attaching to each other along sides (Figure 1*a*). Two *SL* blocks can be arranged into a conjugate pair (Figure 1*b*), with each block as *180* degree rotation on the *y* axis at a corner of the *S* tetracube shared with its counterpart. Various shapes of octocubes have been examined based on the feasibility and flexibility of creating interlocking configurations. *SL* block and its reflective congruent are so far the ones that are most interesting and promising.

(a) an SL block *(b) a conjugate pair*

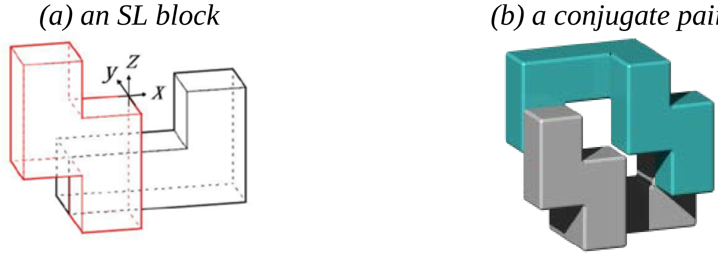


Figure 1: *An SL block and a conjugate pair*

An engagement is defined as the action of inserting a conjugate pair of *SL* blocks based on some geometric transformations. Six types of engagements named as ***h, s, t, d, a, y*** are shown with their corresponding transformations listed by the side of the images in Figure 2. Each image shows the inserted pair of *SL* blocks on the right hand side and its preceding pair on the left. Rotations and translations are represented as $\boldsymbol{R}_{x(angle)}$, $\boldsymbol{R}_{y(angle)}$, $\boldsymbol{R}_{z(angle)}$ and $\boldsymbol{T}_{(x\ y\ z)}$ respectively. The six types of engagements can be divided into two groups of three by whether the engagement consists of 180 degrees rotation along *x* axis or not. Engagements in both groups can be further characterized by 0, -90 and 90 degrees of rotations on *z* axis within the transformation. Translations for engagements are strictly dependent upon rotations.

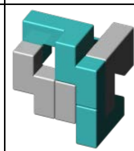
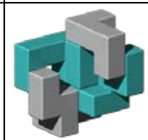
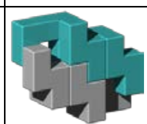
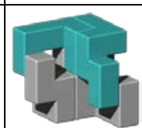
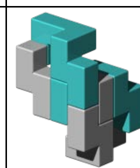
	$\boldsymbol{R}_{z(0)}$	$\boldsymbol{R}_{z(-90)}$	$\boldsymbol{R}_{z(90)}$
$\boldsymbol{R}_{x(180)}$	 $\boldsymbol{R}_{x(180)}\boldsymbol{T}_{(2\ 0\ 0)}$	 $\boldsymbol{R}_{x(180)}\boldsymbol{R}_{z(-90)}\boldsymbol{T}_{(1\ 1\ -1)}$	 $\boldsymbol{R}_{x(180)}\boldsymbol{R}_{z(90)}\boldsymbol{T}_{(1\ -1\ 1)}$
	h *engagement*	***s*** *engagement*	***t*** *engagement*
$\boldsymbol{R}_{x(0)}$	 $\boldsymbol{T}_{(2\ 0\ -1)}$	 $\boldsymbol{R}_{z(-90)}\boldsymbol{T}_{(1\ -1\ 0)}$	 $\boldsymbol{R}_{z(90)}\boldsymbol{T}_{(1\ 1\ -2)}$
	d *engagement*	***a*** *engagement*	***y*** *engagement*

Figure 2: *The six types of engagements and their corresponding transformations*

SL **strands** can be represented as non-commutative multiplications of engagements. Each engagement stands for consecutive actions of making a geometric transformation and then inserting a conjugate pair of *SL* blocks into the strand. Sequences of engagements may lead to the construction of strands that extend and turn orthogonally on the *X-Y* plane and/or shift in the *Z* direction. Some sequences of engagements may result in self-intersecting and unbuildable strands. Periodic strands are strands that interlock one end of the strand into the other, like snakes that bite their own tails. Periodic strands are created if and only if the multiplication of geometric transformations in the sequence of engagements results in the identity transformation. The left hand side of Figure 3 shows the simplest periodic strand $\boldsymbol{a}^4$, which is formed with four consecutive ***a*** engagements. The right hand side of Figure 3 shows some variations of *SL* strands and their generating sequences of engagements. All strands that are not self-intersecting are physically buildable. The building process may not follow exactly the sequence of engagements, but a mapping from the engagement sequence to the building process can be systematically defined.

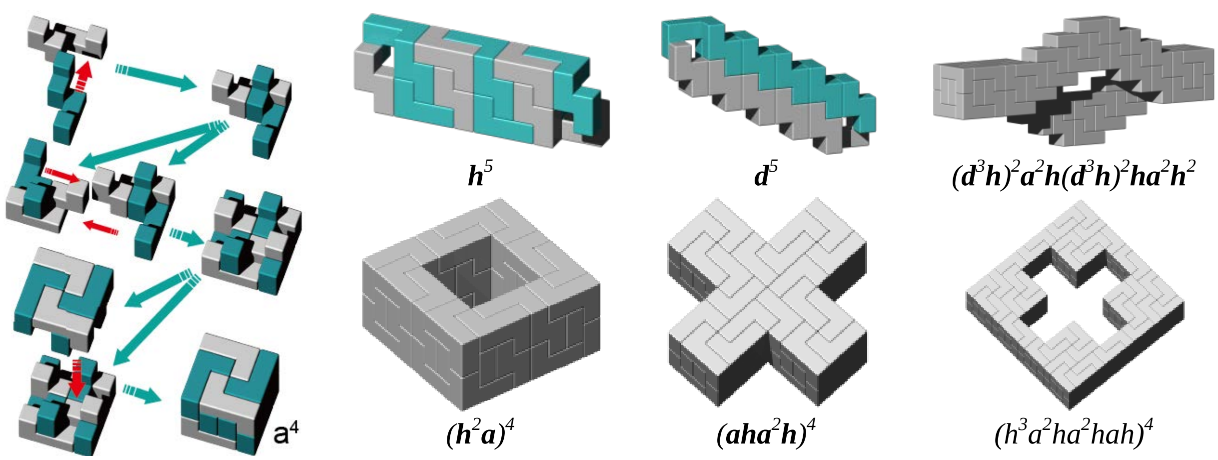


Figure 3: *Variations of SL strands and their engagement sequences*

SL Poly-strand and Polynomial representation

Considering multiplication as concatenation of *SL* engagements and addition as insertion of multiple *SL* strands, polynomial can be used to represent the resulting composite structure called poly-strands, provided that geometric transformations are used as coefficients and transfer corresponding *SL* strands to their designated locations and orientations. Figure 4 shows an *SL* poly-strand that can be defined as the polynomial $(ah^2)^4 + U(ah^2)^4 + U^2(ah^2)^4$, with U stands for $R_{x(90)}T_{(4\ 0\ 0)}$, which transforms the first $(ah^2)^4$ strand to its succeeding strand. The multiplicative identity e represents a null engagement which does nothing. A set that includes geometric transformations, e, *SL* engagements and their inverses can be used as a base to define a non-commutative ring generated by multiplication and addition. Upon the ring, variables, functions and other mathematical constructions can be put on the stage to enrich notations that support the creation, communication and manipulation of sophisticated *SL* constructions.

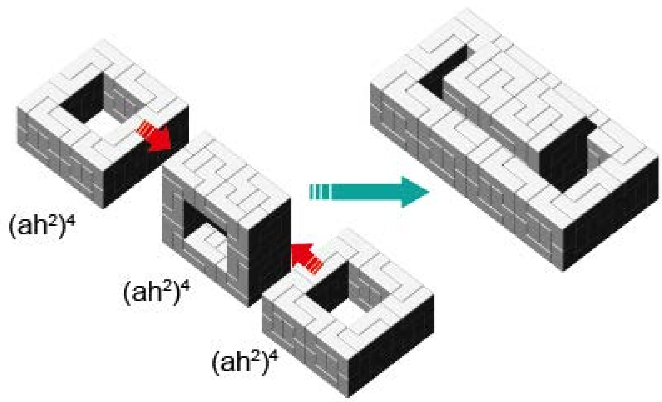


Figure 4: *An SL poly-strand represented with the polynomial* $(ah^2)^4 + U(ah^2)4 + U^2(ah^2)^4$

The function $\textbf{Chain}(n)=(U^n-e)(U-e)^{-1}(ah^2)^4$ with positive integer parameter n defines chain-like *SL* poly-strands with variable lengths. Square frames of variable sizes can be defined with the function $\textbf{Square}(n)=(ah^{2n})^4$. Figure 5 shows a pyramid shaped poly-strand with enclosing square frames defined as $\textbf{Square}(n)$, with n specifies the width of the cavity inside the frame. With V stands for the geometric transformation $T_{(-2\ -2\ -2)}$, the entire poly-strand can be represented as the expression of $\textbf{Square}(0)+V\textbf{Square}(1)+V^2\textbf{Square}(2)+V^3\textbf{Square}(3)$. Pyramid structures of the kind can be generalized as the following function:

$$\textbf{Pyramid}(n) = \sum_{i=0}^{n} V^i \textbf{Square}(i)$$

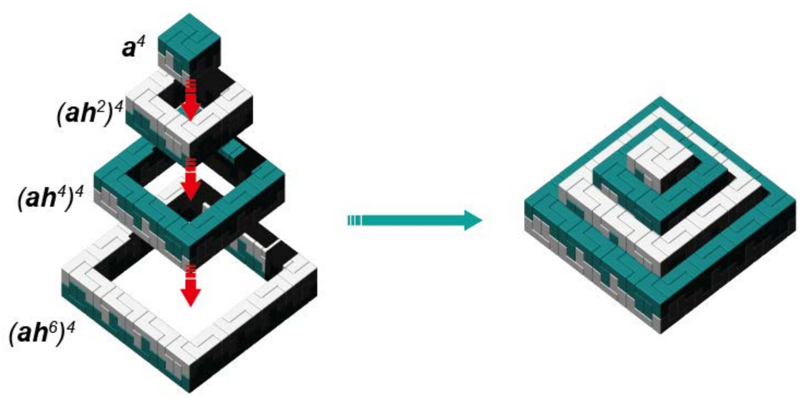


Figure 5: *A pyramid-shaped poly-strand composed of square frames*

Functions may have polynomials as parameters. Figure 6 shows two examples represented by the function ***Pile***$(n, \boldsymbol{w}) = (U^n\text{-}\boldsymbol{e})(U\text{-}\boldsymbol{e})^{-1}\boldsymbol{w}$, with n as the number of stacking layers, ***w*** as the expression that makes up one layer, and U as the transformation $\boldsymbol{T}_{(0\ 0\ 4)}$.

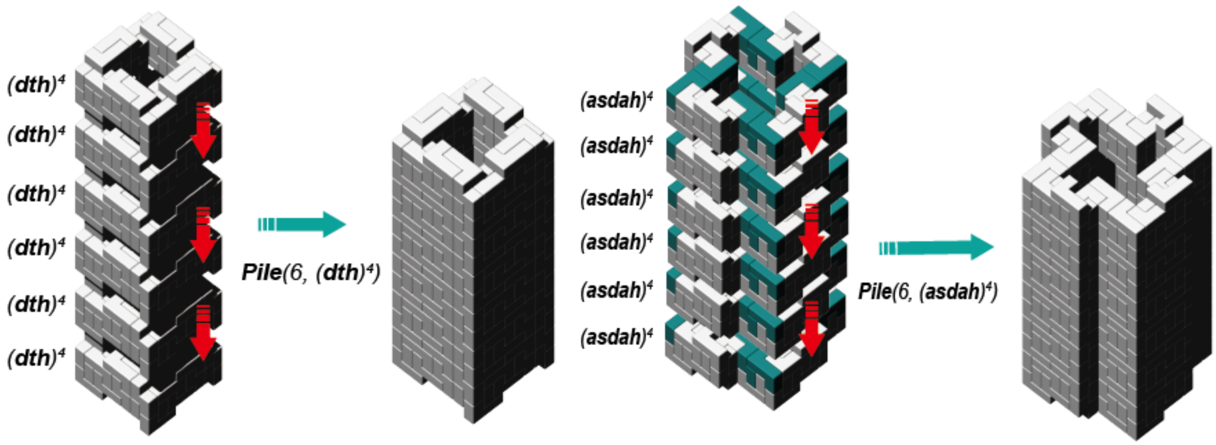


Figure 6: *A function that specifies poly-strands with stacking layers*

Hierarchical definition

Hierarchical definitions may simplify the representation by substituting expressions with variables. For example, with 40 *SL* blocks, one can build a *T*-shaped pentomino (Figure 7). The engagement sequence for the shape is ***hhaahahhhaahhhahaahh***. The sequence can be substituted with the variable ***T*** based on the following hierarchical definition:

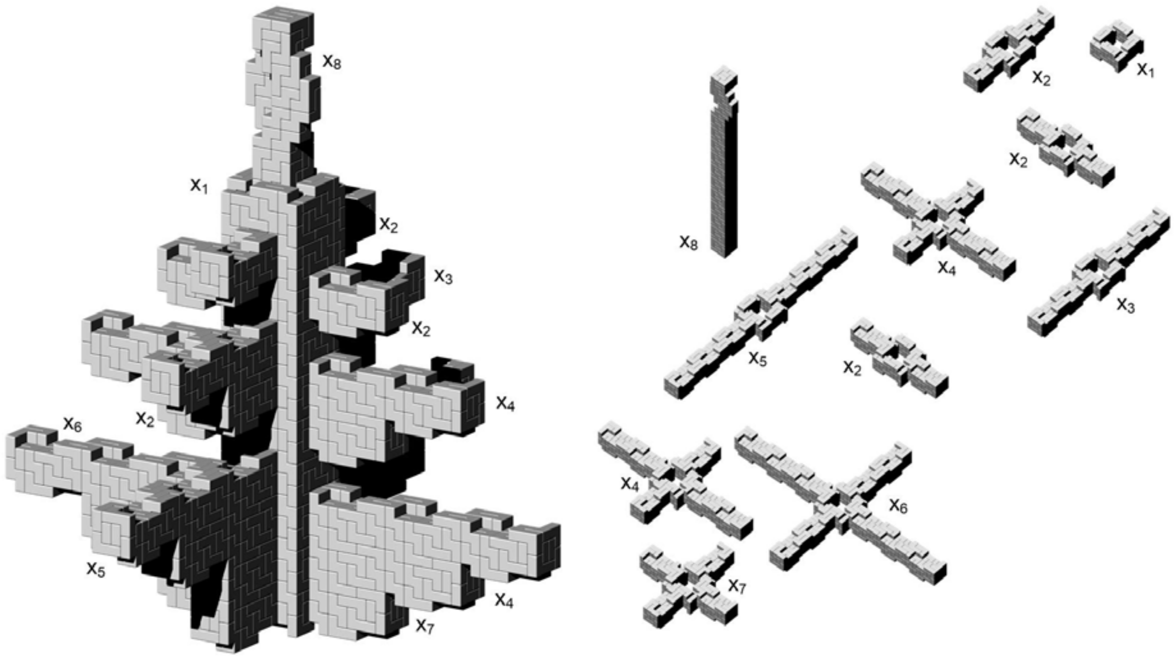
$$\boldsymbol{X} = \boldsymbol{h}\boldsymbol{a}^2\boldsymbol{h}, \text{ and } \boldsymbol{T} = \boldsymbol{h}\boldsymbol{X}\boldsymbol{a}\boldsymbol{h}^2\boldsymbol{X}\boldsymbol{h}^2\boldsymbol{a}\boldsymbol{X}\boldsymbol{h}$$

Variables can explicate features that are otherwise unnoticeable. In this example, the variable ***X*** features three extruded parts of the *T*-shaped structure. It is also recognized that the engagement sequence for ***T*** shape is a palindrome, which reads the same backward as forward. The fact is concurrent with the geometric symmetry of the shape.

$$\boldsymbol{X} = \boldsymbol{h}\boldsymbol{a}^2\boldsymbol{h},$$
$$\boldsymbol{T} = \boldsymbol{h}\boldsymbol{X}\boldsymbol{a}\boldsymbol{h}^2\boldsymbol{X}\boldsymbol{h}^2\boldsymbol{a}\boldsymbol{X}\boldsymbol{h}$$

Figure 7: *The hierarchical definition that simplifies the engagement sequence of the T-shaped strand*

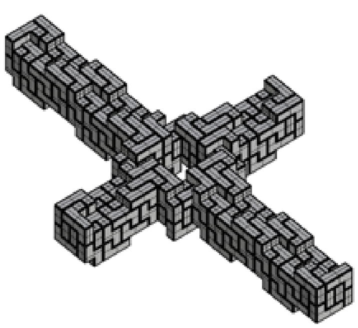
Figure 8*a* shows a tree-like *SL* poly-strand consisting 1024 *SL* blocks. Figure 8*b* shows the 11 *SL* strands that make up the entire tree structure. There are totally 8 types of strands, indexed as X_1 to X_8 for reference. Strands of types X_1 to X_7 are stacked up with the order shown in Figure 8*a*, and the strand of type X_8 is inserted into the vertical cavity at the center of the stack. A computer program implemented with Grasshopper and Python on Rhino was used to parse hierarchical definitions of *SL* poly-strands. Engagement sequences and variable substitutions can be interpreted for automatic model building of poly-strands.



(a) The completed tree. *(b) The 11 parts of the tree.*

Figure 8: *The tree and its compositional strands*

Figure 9 shows the engagement sequence and the hierarchical definition that define $\boldsymbol{X_4}$, which is one of the *SL* strands that make up the tree-like poly-strand $\boldsymbol{X_{tree}}$.



The engagement sequence for strand $\boldsymbol{X_4}$:
ahdhsadhdttahdhdhdhsadhdhdhdttahdhsadhdttahdhdhdhsad hdhdhdtt

Hierarchical definition:
$\boldsymbol{E = hd}$,
$\boldsymbol{F = hsad}$,
$\boldsymbol{f}(n) = \boldsymbol{aE}^{2n-1}\boldsymbol{FE}^{2n-1}\boldsymbol{t}^2$
$\boldsymbol{X_4} = (\boldsymbol{f}(2)\boldsymbol{f}(1))^2$

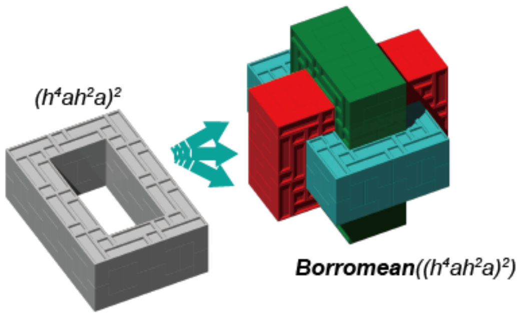
Figure 9: *The engagement sequence and the hierarchical definition for strand* $\boldsymbol{X_4}$

The following hierarchical definition represents the entire tree with the variable $\boldsymbol{X_{tree}}$:
$\boldsymbol{E = hd}$,
$\boldsymbol{F = hsad}$,
$\boldsymbol{G = d^2hd^2}$
$\boldsymbol{f}(n) = \boldsymbol{aE}^{2n-1}\boldsymbol{FE}^{2n-1}\boldsymbol{t}^2$
$\boldsymbol{X_1} = (\boldsymbol{Et})^4$, $\boldsymbol{X_2} = (\boldsymbol{f}(1)\ \boldsymbol{Et})^2$,
$\boldsymbol{X_3} = (\boldsymbol{Etf}(2))^2$, $\boldsymbol{X_4} = (\boldsymbol{f}(2)\boldsymbol{f}(1))^2$,
$\boldsymbol{X_5} = (\boldsymbol{Etf}(3))^2$, $\boldsymbol{X_6} = (\boldsymbol{f}(3)\boldsymbol{f}(2))^2$,
$\boldsymbol{X_7} = \boldsymbol{f}(1)^4$, $\boldsymbol{X_8} = \boldsymbol{a^2h^{21}Ga^2Gh^{21}}$
$\boldsymbol{U} = \boldsymbol{T}_{(0\ 0\ 4)}$,
$\boldsymbol{X_{tree}} = \boldsymbol{R}_{y(90)}\ \boldsymbol{X_8} + \boldsymbol{T}_{(-2\ -2\ 0)}\ (\boldsymbol{X_7} + \boldsymbol{UX_4} + \boldsymbol{U^2X_6} + \boldsymbol{U^3X_5} + \boldsymbol{U^4X_2} + \boldsymbol{U^5X_4} + \boldsymbol{U^6X_3} + \boldsymbol{U^7X_2} + \boldsymbol{R}_{z(90)}\boldsymbol{U^8X_2} + \boldsymbol{U^9X_1})$

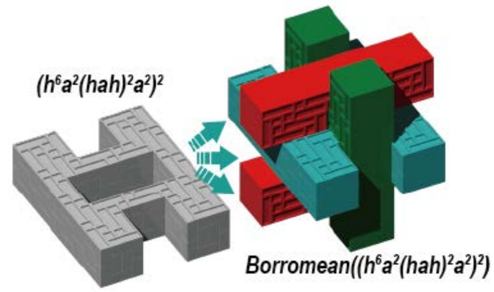
More examples of *SL* Poly-strands

Borromean rings

Borromean rings can be created with three interlocking rings oriented orthogonally (Figure10*a*). Various forms of Borromean rings can be derived by using periodic strands each encloses a void just large enough to hold the other strands such as the structures shown in the left hand sides of Figure 10*a* and 10*b*. The function can be defined as $Borromean(x) = T_{center(x)}(\ x + R_{x(90)}R_{z(90)}x + R_{y(90)}R_{z(90)}x)$, with x as the basic strand for the ring and $T_{center(x)}$ as the translation that takes x to locate its centre at the origin of the coordinate system. Figure 11 shows more examples with photos of physical models of structures that consist of Borromean rings.



(a) The Borromean rings with rectangular strands



(b) The Borromean rings with H-shaped strands

Figure10: *Borromean rings*

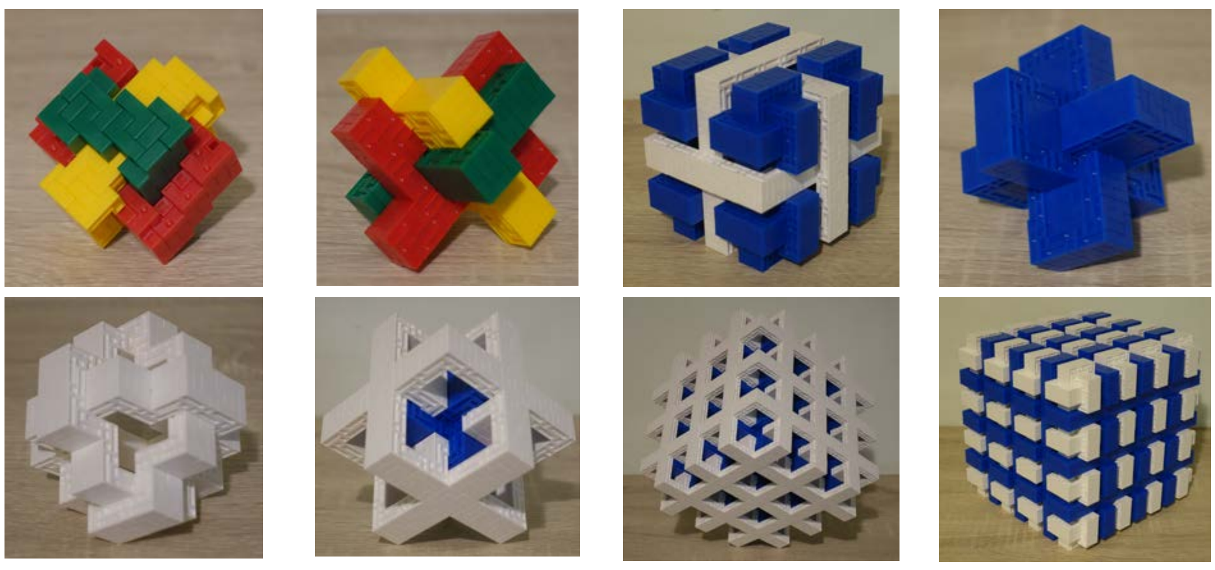
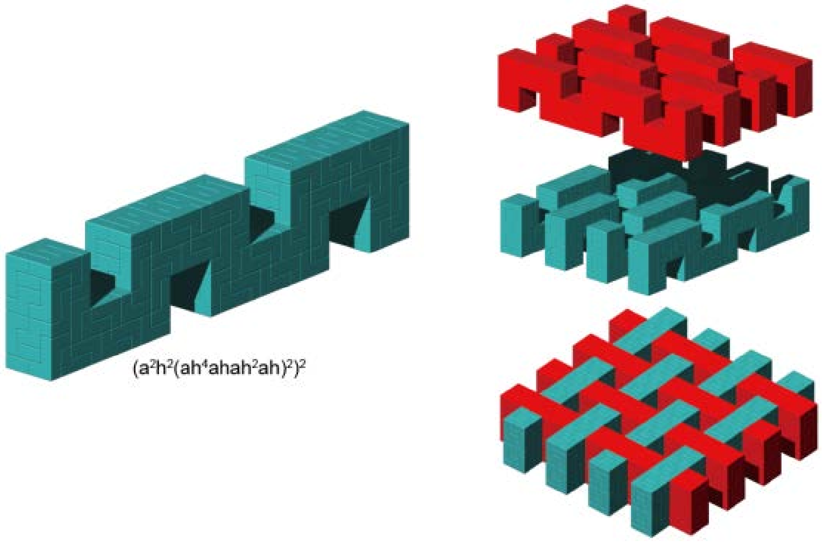


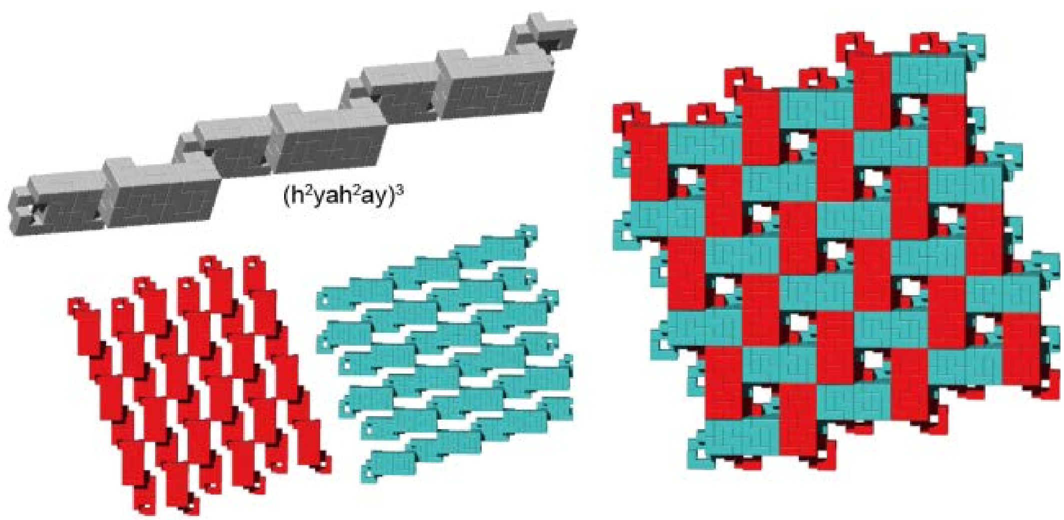
Figure11: *More examples of Borromean rings*

Interwoven Panel

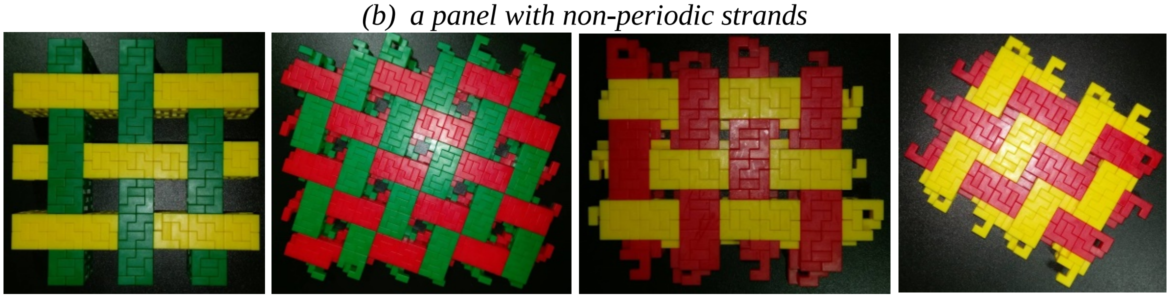
An interwoven panel consists of two groups of parallel *SL* strands that are perpendicular to each other. Every strand is locked within the configuration to form a strong structure. The assembly of such interwoven panels is difficult but possible. Within the interlocked panel, only some *SL* blocks on the periphery can be removed without being topologically blocked. No internal *SL* blocks can be removed without breaking other blocks. The panel can be extended in four directions. Figure 12*a* shows a panel made with periodic strands of 8 cubes in thickness. Figure 12*b* shows a panel consists of non-periodic strands, which is half as thick as the panel in Figure 12*a*. Figure 12*c* shows four photos of physically built interwoven panels of various configurations.



(a) a panel with periodic strands



(b) a panel with non-periodic strands



(c) Photos of interwoven panels with different configurations

Figure 12: *Examples of interwoven panels*

Discussion

With 6 types of engagements, *SL* blocks can be used to create strands that may extend and turn orthogonally on the *X-Y* plane and shift in the *Z* direction. Multiple strands can be composed to form more sophisticated structures such as Borromean rings and interwoven panels. The relationship between arts and mathematics can be illustrated with the relationship between literature and the grammar which the language of the literature is based upon. A language cannot derive rich expressiveness for artistic creation without a coherent syntactical structure. It is expected that mathematical constructions such as polynomial, function and hierarchical definition of *SL* poly-strands may serve as means to encode self-interlocking structure of *SL* blocks, and to constitute a coherent and expressive language for artistic creations. The achievement so far is minor, but a vision towards a wide world worth of exploration has been opened.

Acknowledgements

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