Magnetic Sphere Constructions

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Abstract

We investigate constructions made from magnetic spheres. We give heuristic rules for making stable constructions of polyhedra and planar tilings from loops and saddles of magnetic spheres, and give a theoretical restriction on possible configurations, derived from the Poincaré-Hopf theorem. Based on our heuristic rules, we build relatively stable new planar tilings, and, with the aid of a 3D printed scaffold, a construction of the buckyball. From our restriction, we argue that the dodecahedron is probably impossible to construct. We finish with a simplified physical model, within which we show that a hexagonal loop is in static equilibrium.

Introduction

Magnetic spheres are a popular desk toy that were often sold under the trade names *Buckyballs*, *Neocube*, *Nanodots* or *Zen Magnets*, until their sale as a toy was banned in many countries for safety reasons. They are often used to construct polyhedra and other geometric objects. See Figure 1.[1]

One could build a model of a polyhedron from, say, wooden spheres, glued together in such a way that there is a sphere corresponding to each vertex of the polyhedron. However, our spheres are magnets, and we want the magnetic forces to hold the structure together, with no glue required. Part of the aesthetic appeal of these magnetic spheres is that no other methods of attachment are required to build beautiful sculptures. So, in addition to the locations of the spheres, we also need to specify the orientation of the magnets – which way the poles point. In this paper, we investigate the following question:

Which polyhedra are possible to construct from magnetic spheres?

Figure 1: *The tetrahedron, octahedron, cube and icosahedron*

To be precise about what we mean by "construct a polyhedron" we introduce the following definition.

Definition. To any configuration of magnetic spheres we associate the *incidence graph*, which is a graph embedded in three-dimensional space with a vertex corresponding to each magnetic sphere, and an edge between two vertices whose corresponding spheres are in contact. We say that we have a *construction of a*

[1] For many more examples, see http://www.flickr.com/groups/magnetspheres.

polyhedron from magnetic spheres when the polyhedron is the incidence graph of a stable configuration of magnetic spheres.

In the sense of this definition, Figure 1 shows constructions of all but one of the platonic solids. To the best of our knowledge, nobody has been able to construct a dodecahedron. Note that the construction in Figure 2 is not a dodecahedron in our terminology; rather this is a *cantellated* dodecahedron, also known as the *rhombicosidodecahedron*. *Cantellation* shaves off every edge and vertex of a polyhedron, producing a quadrilateral where each edge of the original polyhedron was, a new face where every vertex of the original polyhedron was, and shrinking (but not eliminating) the faces of the original polyhedron.

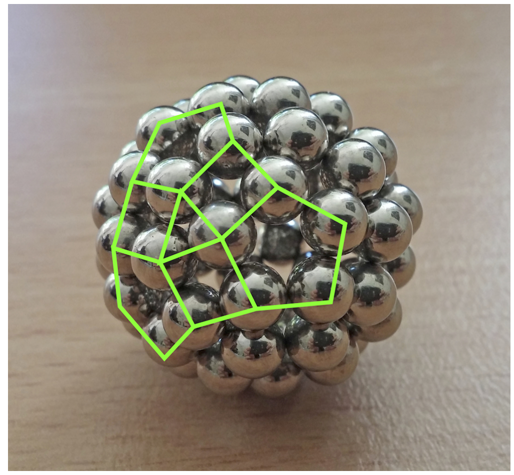
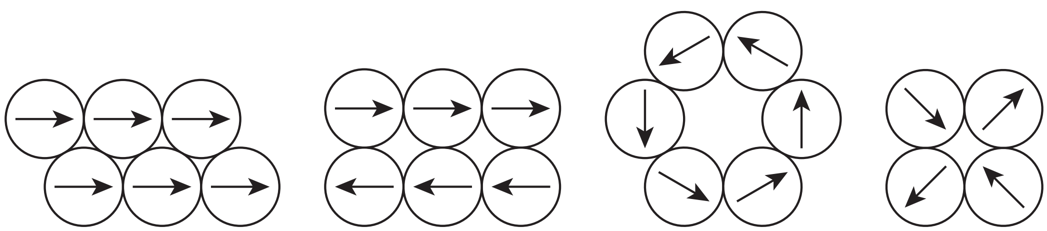


Figure 2: *A cantellated dodecahedron construction, and part of its incidence graph.*

Some Basic Constructions

A line of magnets can be built, with the magnets all pointing in the same direction, sitting head to tail. Two lines can be joined parallel to each other in two ways; pointing in the same direction, or in opposite directions. See Figures 3a and 3b. Two lines are *staggered* if they have been joined parallel with their poles pointing in the same direction, making a row of alternating triangles. Two lines are *in step* if they are joined with their poles pointing in opposite directions, making a row of squares. A *loop* of magnets is a polygon in which each magnet is aligned head to tail. A *saddle* of magnets is a polygon in which magnets alternate between pointing into the polygon and out of it. A saddle must therefore have an even number of magnets in it.

Note that loops, and saddles in particular are somewhat fuzzy concepts: the directions of the magnetic poles relative to a polygon can be altered to an extent, depending on the surrounding magnets, while still playing a similar role in a structure.



(a) *Staggered.* (b) *In-step.* (c) *Hexagonal loop.* (d) *Square saddle.*

Figure 3: *Basic constructions.*

Loops and Saddles

Generally, people combine staggered and in-step strips, and loops together to build larger constructions such as those shown in Figure 4. In this paper we only consider structures where the centers of the magnets lie on a sphere or a plane. The interactions between magnets in denser structures are more complicated. Constructions involving staggered and in-step strips also seem to be difficult to analyse, so we put these aside, and concentrate on constructions made from loops.

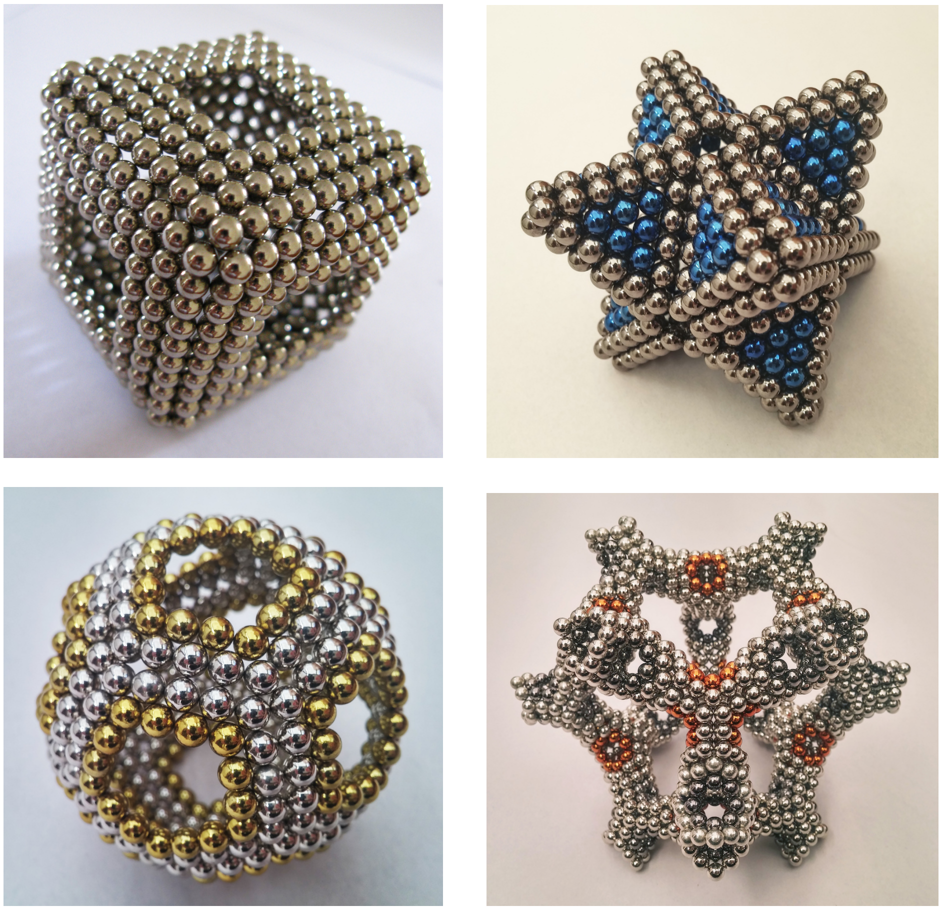


Figure 4: *A selection of sculptures built from magnetic spheres.*

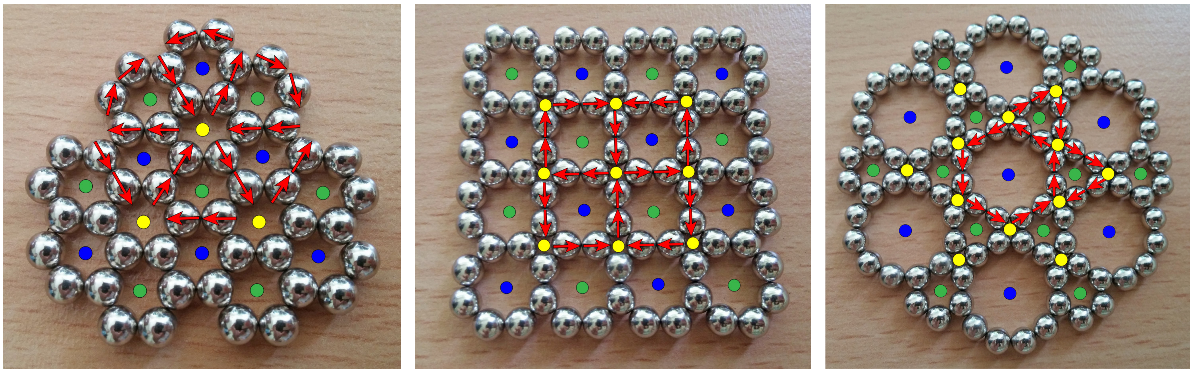
The cantellated dodecahedron shown in Figure 2 is made from 12 pentagonal loops, both conceptually and physically: an easy way to build it is to make the pentagonal loops first and then put them together. Note that the cantellated dodecahedron also has 20 triangular loops, and 30 square saddles. In principle then, it should be possible (but much more difficult) to build the cantellated dodecahedron by making the 20 triangular loops first, then putting them together. Square saddles are not stable on their own: they immediately deform to become square loops.

We propose the following heuristic rules for making stable polyhedral structures and planar tilings:

1. Loops of any size are stable.
2. Square saddles are stable when supported by neighbouring loops.
3. Hexagonal saddles are less stable, and higher order saddles even less so. (In fact, we have no examples of stable structures with higher order saddles!)

The construction method of combining pre-made loops is a simple and reliable method to build surface-like constructions, but it precludes other interesting possibilities. We introduce a number of apparently new planar designs which use loops and saddles, but which cannot be made entirely from pre-made loops. See Figure 5.

The hexagonal tiling in Figure 5a has hexagonal loops in both directions, and hexagonal saddles. This structure is quite fragile – in fact it seems difficult to build the tiling any further out than as shown in the



(a) *Hexagonal tiling.* (b) *(4,8,8) tiling.* (c) *(4,6,12) tiling.*

Figure 5: *New planar designs. The red arrows show the directions of the poles of the magnets, blue dots are anticlockwise loops, green dots are clockwise loops, and yellow dots are saddles.*

figure. The tiling of octagons and squares in Figure 5b is more stable, but is still challenging to build. The tiling in Figure 5c can be made from loops alone, although the section of the pattern shown, as opposed to the infinite tiling, cannot. We'll return to a non-planar, polyhedral design (the buckyball) in a later section, after developing some theory.

A Theoretical Restriction on Loops and Saddles

Let's say we wish to build a polyhedron by first selecting the polygons we wish to use (which will be either loops or saddles), and degree of the vertices. We know that the Euler Characteristic equation

$$\chi = v - e + f = 2 \tag{1}$$

must hold, where v, e and f are the numbers of vertices, edges and faces respectively. There is however another general rule that must be satisfied, coming from considering the magnetic field over the surface of the polyhedron.

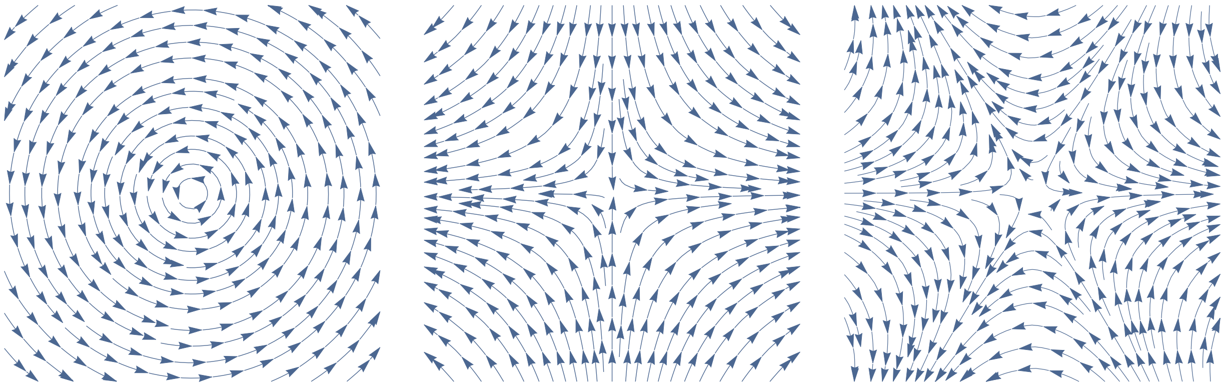
Definition. Let a vector field V be defined on a differentiable surface M. Let x be an isolated zero of V, and D a closed disk centred at x, such that x is the only zero of V in D. The *index* of V at x, $\text{index}_x(V)$, is the degree of the map $u : \partial D \to S^1$ from the boundary of D to the circle given by $u(z) = \frac{V(z)}{|V(z)|}$.

In other words, one walks around the zero of the vector field, keeping track of which way the vector field is pointing. The index is the count of the number of times (with sign) that the vector field turns in the same direction as we are walking around the zero. Figure 6 shows examples of vector fields in the plane, that have zeroes with various index values.

Theorem (Poincaré-Hopf [3, p134])**.** *Let M be a compact orientable differentiable surface. Let V be a vector field on M with isolated zeroes (at) x_i. If M has boundary, then M points in the outward normal direction along the boundary. Let $\chi(M)$ be the Euler Characteristic of M. Then*

$$\sum_i \text{index}_{x_i}(V) = \chi(M).$$

From the Poincaré-Hopf theorem we can add another general restriction on what kinds of loops and saddles can be used to tile a polyhedron made from magnetic spheres. The magnets induce a vector field on a



(a) *A zero with index* 1. (b) *A zero with index* -1. (c) *A zero with index* -2.

Figure 6: *Example indices of zeroes of vector fields.*

surface that passes through their centers. In practice, this vector field is very complicated – in the appendix we look at a simplified physical model. However, under certain conditions we may consider an even simpler vector field that helps us understand the possibilities. We construct our simplified vector field guided by Figure 6: in each loop of magnets we put a zero of index $+1$ at its center, in each square saddle a zero of index -1 at its center, in each hexagonal saddle a zero of index -2 at its center, and in general in each $2n$-gon saddle a zero of index $1-n$ at its center. We extend our vector field to cover the surface our polyhedron is built on with no further zeroes. Then the Poincaré-Hopf theorem gives a further restriction on the number and kinds of loops and saddles that we can use.

(a) *Truncated octahedron.*

(b) *Truncated cuboctahedron.*

Figure 7: *Polyhedra made from loops and saddles.*

For example, the cantellated dodecahedron in Figure 2 has 12 pentagonal loops, 20 triangular loops, and 30 square saddles, so the indices add up to $(12+20)(+1)+30(-1)=2$, which is the Euler characteristic of the sphere. In general any polyhedron with unit edge lengths can be cantellated to produce a new polyhedron that can be made from spherical magnets: the faces and vertices of the original polyhedron become loops (in opposite directions), and the edges become square saddles. The Poincaré-Hopf condition for the cantellated polyhedron is satisfied because it is the same equation as the Euler characteristic for the original polyhedron.

Figures 7a and 7b show the truncated octahedron (8 hexagonal loops and 6 square saddles) and the truncated cuboctahedron (6 octagonal loops, 8 hexagonal loops and 12 square saddles). Neither of these are cantellations of simpler polyhedra, although they are closely related to cantellated polyhedra.

The platonic solids, as shown in Figure 1, don't fit into our system of loops and saddles very well; rather they are made from staggered and in-step strips, with the exception of the tetrahedron, which is too small for any of these two-dimensional analyses to apply. The cube can be made from a rolled up four-long and two-high in-step strip, and the octahedron can be made from a rolled up three-long and two-high staggered strip. The icosahedron can be made by rolling up a five-long and two-high staggered strip into a cylinder, forming ten of the twelve vertices, then putting the two polar magnetic spheres in place with no real thought as to their orientations. All of these are stable because they are made from the very stable staggered and in-step strips. In contrast, the dodecahedron has pentagonal faces so cannot be made from staggered or in-step strips, and it also cannot be built from loops and saddles: there can be no saddles at all since no face has an even number of sides. By the Poincaré-Hopf theorem, if we have no saddles then we can only have two loops, not the 12 that are needed for the dodecahedron. One could try having two loops forming opposite faces of the dodecahedron, with the remaining ten equatorial pentagons having magnets aligned so that they all point in the same direction around the sphere. (If the loop pentagons are near the north and south poles of the sphere, then we would have all magnets pointing east, say.) However, these equatorial pentagons are not stable, and the structure collapses. This leads us to suspect that a dodecahedron is impossible to build.

Note that the Poincaré-Hopf theorem can be made to apply to planar tilings, even though the plane is not a compact surface. If we take a repeating unit of a planar tiling, then we can (abstractly) roll it up to form a torus, which then has Euler characteristic zero, and we can apply the same analysis as for the spherical case.

The Buckyball

We spent a significant amount of time trying to build a truncated icosahedron (also known as a buckyball) from spherical magnets. Here the faces of the polyhedron consist of 12 pentagons and 20 hexagons. The pentagons must be loops since they cannot be saddles, and then this determines the orientation of all 60 magnetic spheres, up to deciding whether the pentagons are oriented clockwise or anticlockwise on the sphere. However, neighbouring loops that have opposite orientation meet at chains of magnets that are in staggered formation, whereas loops that have the same orientation meet at chains of magnets that are in-step (see Figures 3a and 3b). That is, locally two pentagonal loops with opposite orientations will not stably touch at only one point, as required in the buckyball. So, all loops must have the same orientation.

In the Poincaré-Hopf theorem, the pentagons contribute $12(+1)$ to the sum. This suggests that half of the hexagons should be saddles, contributing $10(-2)$ and half of the hexagons should be loops, contributing $10(+1)$. This is similar to the hexagonal tiling in Figure 5a, where a third of the hexagons are clockwise loops, a third are anticlockwise loops, and the remaining third are saddles. However, on the buckyball, there is no similar symmetric way to split the hexagons into two subsets.

Figure 8: *A 3D printed scaffold, and a buckyball of buckyballs arranged upon it.*

Nevertheless, with a 3D printed scaffold on which to build the buckyball, we were able to place 12 pentagonal loops and form a somewhat stable structure, see Figure 8. Given our experiences with how fragile the structure is, we don't think it would survive without the scaffold.[2] This example doesn't fit terribly well into our loops and saddles model. Perhaps one could accommodate it by positing that in addition to a saddle at the center of each hexagon, there is a loop at the midpoint of each edge between two hexagons. Then we get $12(+1)+20(-2)+30(+1)=2$. However, it might be better to see the buckyball as an edge case, on the boundary of the set of tilings well described by our model.

References

[1] Tai L. Chow. *Introduction to electromagnetic theory: a modern perspective*. Jones & Bartlett, 2006.

[2] Samuel Earnshaw. On the nature of the molecular forces which regulate the constitution of the luminiferous ether. *Trans. Camb. Phil. Soc.*, 7:97–112, 1842.

[3] Victor Guillemin and Alan Pollack. *Differential Topology*. AMS Chelsea Publishing. AMS, 2010.

[4] Heinz E. Knoepfel. *Magnetic Fields: A Comprehensive Theoretical Treatise for Practical Use*. Wiley, 2008.

A Physical Model

In this appendix, we describe a model of the physical interactions between the magnets, and work out some details for a planar example. This model is itself a simplified approximation of the full physical definition.

Definition. The *moment*, $\vec{m}$, of a magnet is a vector whose magnitude represents the strength of the magnet, and whose direction points from pole to pole (conventionally, south to north).

The moment of a magnet determines how that magnet interacts with other magnetic fields. In our case, we assume that each magnet is identical, and therefore of equal strength. Since each sphere is a permanent magnet, we assume that the magnetisation is independent of applied magnetic fields. Outside of the sphere, the magnetic field can be treated as that of a dipole and inside it is in the direction of the moment [4, p70].

Definition. Let μ_0 be the permeability of free space (a physical constant), $\vec{r}$ the position vector (relative to the dipole), $\vec{m}$ the moment. Then the magnetic field, $\vec{B}$, of a dipole is given by

$$\vec{B}(r) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m}\cdot\vec{r})}{|\vec{r}|^5} - \frac{\vec{m}}{|\vec{r}|^3}\right) \tag{2}$$

(See [1, p149].) Note that the field dies off quickly with $|\vec{r}|$, which allows us to view configurations as a local problem; generally one can consider adjacent magnets as affecting each other, while more distant magnets are insignificant. The magnetic fields add with simple vector addition.

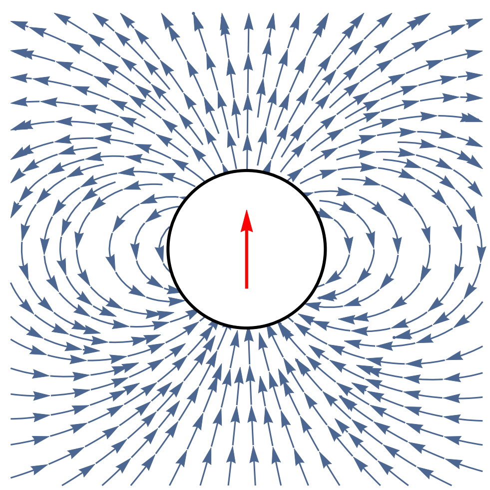
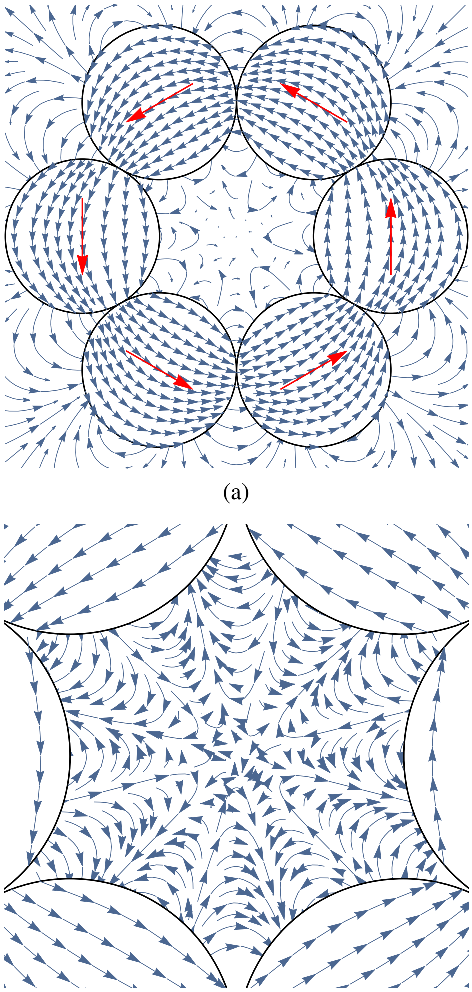


Figure 9: *A horizontal slice of the external field of one magnet with moment $\vec{m}=(0,1)$.*

Example (Hexagonal loop)**.** The magnetic field for a hexagonal loop (or rather, a planar slice of the field) is shown in Figure 10. Note that there is not a zero of index $+1$ at the center, as we might have predicted. Instead, there is a zero of index -5, but there also seems to be six zeroes of index $+1$ near the equators of the magnets. The sum of the indices within the loop is then $+1$, the same as in our simplified model.

[2] Also see https://www.youtube.com/watch?v=DXnEbyo77wI for a video of this structure.



(a)

(b)

Figure 10: *Magnetic field from a hexagonal loop on the plane of the loop. Inside each magnet, only the total of the contributions to the magnetic field from all other magnets is shown. Outside of the circles, the total contribution from all magnets is shown. Figure 10b shows a closeup view of the center of the loop.*

A magnet in a magnetic field will experience a rotational force (i.e. torque) that attempts to align it with the field. We assume that the magnet is acted on by the magnetic field as if the magnet were a single dipole.

Definition. The *torque*, $\vec{\tau}$, exerted by a field $\vec{B}_2$ on a dipole moment $\vec{m}_1$ is

$$\vec{\tau} = \vec{m}_1 \times \vec{B}_2 \tag{3}$$

The torque vector gives the axis about which the magnet will rotate, in the direction given by the right hand rule. Hence the torque is zero if the moment is aligned with the field, and nonzero otherwise. In our case, $\vec{B}_2$ is the net magnetic field from all other dipole moments except $\vec{m}_1$. A magnet's own field does not act on itself.

Assuming that the positions of the magnets are fixed, we say that a configuration is in *rotational equilibrium* if the torque on each magnet induced by the net magnetic field of the other magnets is zero. In our hexagonal loop example, we see that the red arrows of the moments of the individual magnets are aligned with the field from the other magnets, so there is no torque on the magnets, and, assuming that the positions of the magnets are fixed, the configuration is in rotational equilibrium.

For any set of positions of n magnets, there is at least one configuration with these positions which is in rotational equilibrium. We imagine the magnets held in place, but free to rotate. The space of possible configurations with the given magnet directions is $(S^2)^n$, which is a compact space. The potential energy of the system is a continuous function on this space, which is therefore bounded and so has a minimum, at which the configuration will be in rotational equilibrium.

Definition. The *force* $\vec{F}$ exerted by a field $\vec{B}_2$ on a dipole moment $\vec{m}_1$ is

$$\vec{F} = \nabla(\vec{m}_1 \cdot \vec{B}_2) \tag{4}$$

A configuration is in *static equilibrium* if the external torques and the external forces acting on each magnet both sum to zero. To fully understand which configurations of magnets can be built as sculptures, we would additionally need *stable equilibrium*. That is, that the arrangement of magnets returns to the configuration after small disturbances. Earnshaw's Theorem [2] rules out stable equilibria considering only the magnetic forces, so we must at minimum also consider *normal contact forces*. The normal contact force acts on any two spheres in contact and prevents them from interpenetrating. In general, we would also need to pay attention to the effects of friction and gravity, and their strengths relative to each other and the magnetic forces.

When we calculate the magnetic forces on the magnets in the hexagonal loop in Figure 10, we find that they all point towards the center of the loop. In this case, the normal contact forces between the spheres counteract the magnetic forces, so we conclude that the hexagonal loop is in static equilibrium, without requiring friction forces. Showing that the configuration is in stable equilibrium is however beyond the scope of this work.