Thinking Visually: Triangles as Units of Area

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Abstract

Much mathematical education emphasises verbal communication, either teacher exposition or possibly group discussion. Practical activities are often presented as “how to” instructions, and although they may have significant potential for mathematical exploration the mathematics can easily become secondary. At a more advanced stage the emphasis shifts to algebraic methods, which are generally so powerful that more visual approaches may be undervalued. Although there are some resources available that are designed to stimulate a visual approach to mathematics more would be welcome. The workshop will present an opportunity to explore some ideas and challenges based around the concept of area with particular (but not exclusive) reference to triangles. The challenges vary in difficulty: some could be used in primary education, others are more demanding.

Visual Mathematics

Over one hundred years ago Poincaré made the distinction between two kinds of (mathematical) minds: logicians, who prefer to treat problems by analysis, and intuitionists, who prefer geometry [1]. The style of thinking preferred by geometers is summarised in Einstein's response to Hadamard's questions [2]:

> The above mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will.

Archaeological evidence seems to suggest that visual, geometrical thinking is more fundamental than analytical thinking, but it could be misleading since physical ornament and cave paintings survive whereas, by their nature, thoughts and arguments do not, and it is difficult to imagine how evidence of analytical thinking might remain. It is equally difficult to conclude that the production of geometrical artefacts implies mathematical thought. The problem is nicely summarised in the guidelines for Bridges Workshops: It would be great to come away from the workshop being able to answer "what did you learn today?" not just "what did you make today?".

Ethnographic evidence provides little help. It is certainly the case that traditional crafts have mathematical implications, for example Paulus Gerdes has analysed the geometrical ideas in the work of African craftsmen, and suggested ways to use African designs to learn about the theorem of Pythagoras in the classroom [3], but it does not follow that the craftsman know the theorem (although they might). Similarly traditional patterns can be analysed using the mathematics of symmetry groups, but, as Grünbaum states, "The main problem is with the very idea of symmetry. There is no basis whatsoever to assume that symmetry-- an isometric mapping of the ornament onto itself--was anywhere or at any time motivating artists or craftsmen." [4] The only reliable evidence of mathematical thinking, analytical or geometric, comes from the historical record of many cultures [5], and both analytic (generally arithmetical) and geometric thinking seem to be equally in evidence.

In discussing mathematical understanding Poincaré identifies a difference between a reasoned argument and a demonstration that relies on a sequence of images, commenting, "These [learners] often deceive themselves: they do not listen to the reasoning, they look at the figures; they imagine that they have understood when they have only seen." [1] The problem is that visual perception is almost immediate and is processed quickly, and there is evidence that logical, analytical thinking is demanding, and we generally avoid it wherever possible [6]. "One picture is worth one thousand words", and visual presentations provide a powerful means of communication, but their strength is also a weakness: it is surprisingly difficult to *think* visually.

What if … ?

One technique that helps to stimulate thinking is to consider what happens when something is changed. It is quite widely applicable, but in a mathematical context it can range from changing one parameter in the statement of a problem (often used in the context of a classroom investigation, where the outcome is sometimes nothing more than spotting a pattern), through changing the problem (among the problem-solving techniques suggested by Polya [7]), to a more radical change of the rules (for example non-Euclidean geometry). It has the advantage that there is no difficulty in stating what the change is [8,9], and it is quite easy to come up with stereotyped suggestions (What if equality is changed to inequality? What happens in three dimensions? What if we use triangles instead of squares? …), but can often provide fruitful lines of enquiry. There is often a temptation to suggest questions that have obvious answers, which does not usually help much, and might need to be discouraged in a classroom context.

Dynamic geometry packages have made it easier to investigate "What if … ?" questions, with an added advantage that standard methods of construction quickly become second nature. There is a danger, however, that motivation to prove an assertion is lost. It becomes so obvious that some property (for example an incidence of three lines) is conserved when the construction is changed that it is difficult to imagine that it could be otherwise. Incidentally, imagining how something could be otherwise is a useful variation on the "What if …?" tactic. Seeking a proof is important in an educational context because of the additional insight it provides, and it is another way to develop facility in thinking visually.

The Problem of Communication

Einstein's response expresses a difficulty with visual thinking that is less evident with algebraic thinking, still less with verbal thinking. It is difficult to express images in "conventional words or other signs". There is even some evidence that visual and verbal thinking are mutually exclusive. In the preface to *Drawing on the Right Side of the Brain*, Betty Edwards reports that she sometimes stops talking while demonstrating drawing in her classes, and then finds it difficult to resume. Wherever visual thinking might be located in the brain it is difficult to keep up a commentary on it. The implication is that group work, which is widely used in mathematics classrooms, could actively hinder visual thinking, and there is a need to find ways in which learners can share their visual ideas. An obvious approach is to encourage sketching, and the use of physical materials can help.

Modern technology has made it much easier for lecturers to present visual ideas, but it does not follow that visual thinking is any easier for their students. In fact the danger expressed by Poincaré is even more immediate: it is so much easier to see what is being communicated that there is less need to think. Usually the images or animations in presentations require significant time to create, and such techniques would not be suitable in informal group investigations. Any innovation that helps learners to exchange (rather than investigate) visual ideas quickly and easily would be welcome.

Proof

A useful resource to stimulate visual thinking is the *Proofs without Words* series [10, 11, 12]. Since there are (usually) no words it is necessary to think in a purely visual way in order to understand what is presented. The introductions to these books discuss whether the contents actually constitute proofs in the mathematical sense. The twentieth century emphasis on algebra and logic, and the well-known fallacies that depend on misleading diagrams [13], have resulted in a general distrust of visual demonstrations, and most mathematicians would agree that the contents do not constitute proofs. There is another view, however, and it can be argued that the proofs are valid provided that they are general. It does not matter that the conclusion can be *seen*, but that a valid relationship is communicated. Translating the relationship into conventional symbols might be laborious, but not impossible, and could be a useful exercise to develop further insight.

The Workshop

The workshop will take as its starting point a simple geometric construction that occurs in Plato's account of Socratic dialogue in *Meno*. A square has its corners at the mid-points of the sides of a larger square, and Socrates leads a slave to the conclusion that the smaller square is exactly half the area of the larger one. Max Bill has used this arrangement, first in *The Red Square*, 1946, and explored many of its possibilities, in particular with respect to colour, in the 1970s [14]. Participants will be invited to suggest ways to change the construction, in particular from squares to triangles, and to consider the consequences. It is not possible to predict exactly what will happen, but some fairly obvious ideas (prompted if necessary) could lead to considerations of:

- the measurement of area
- Goldberg polyhedra
- the theorem of Pythagoras
- other theorems about areas
- tilings in two and maybe three dimensions.

Participants can expect opportunities to think about geometry in a visual (as opposed to verbal or algebraic) way, and, depending on their interest, might take away:

- suggestions for classroom investigations
- a way of working with learners that encourages them to pose their own questions
- a way of working with learners that stimulates creativity
- new insights into some simple geometry
- some thoughts about the nature of proof
- ideas that can be used in art/craft works.

Wider Importance

Most learners will not become academic mathematicians but the subject is so important that its study is compulsory in most countries. Verbal facility is associated with power: laws and contracts consist of words, and the legislature in the UK is called parliament, literally a place where people speak; the visual is not associated with the exercise of power, and it is often undervalued. In addition thinking visually, as opposed to seeing, is difficult, it can be hard to communicate, and it is often slow. For all of these reasons little time is devoted in the curriculum to the education of vision, and for most people what could be a significant ability remains undeveloped. Mathematics classes provide one context where it could receive more attention.

References

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