Introducing the Kasparian Constructions

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Abstract

This paper serves as an introduction to a type of geometric construction discovered by Raffi J. Kasparian while writing a polyhedral construction tool, named "Archimedean." Archimedean's unique exploratory approach to polyhedral construction led to the discovery of several objects where the faces of the polyhedron traverse the surface more than once before joining. Alice Petillo has used the Archimedean program in teaching geometry to undergraduate students at Marymount University in Arlington, Virginia for the last several years. Kasparian and Petillo have collaborated to bring this discovery to the larger mathematical community. Archimedean may be found at http://www.quantimegroup.com/solutions/pages/Archimedean2.0/Archimedean.html.

Introduction

Polyhedra have delighted and fascinated mathematicians and artists for millennia, yet almost 2000 years passed from the time the Platonic and Archimedean Solids were discovered until new solids were discovered. Did 2000 years of mathematicians really just miss them? No. New solids were "discovered" when previously accepted necessary conditions for polyhedral construction were abandoned. When Kepler assembled star polygons into surfaces, he revised two common notions by 1) allowing polygons with self-intersecting edges and 2) allowing faces to intersect each other. And when Poinsot allowed corners to be formed by wrapping faces more than once around a vertex, he was breaking another barrier. These three changes enabled the discovery of many more polyhedral constructions with identical corners. This led in time to the completion of the set of uniform polyhedra, which includes 75 non-prismatic polyhedra and an infinite number of prisms and anti-prisms. One additional construction known as Skilling's figure is a poor relative, sometimes admitted to the set of uniform polyhedra, but considered to be degenerate because it involves coinciding edges where four polygonal faces meet.

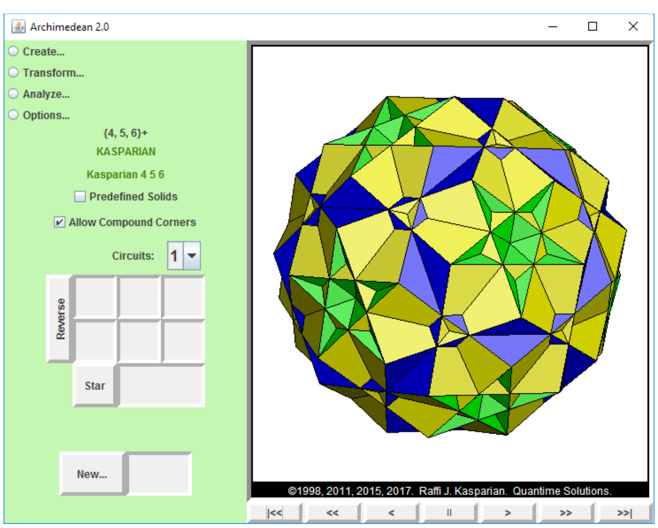


Figure 1: *Screenshot of Archimedean 2.0 showing K7.*

After encountering "Shapes, Space and Symmetry" by Alan Holden [1], I, Raffi J. Kasparian, was seized with the desire to construct the known polyhedra, to transform them, and to analyze them myself in order to gain a greater understanding of their nature. Since my expertise is in programming, I wrote my own 3-D construction and rendering tool, which I called "Archimedean". It approached polyhedral construction from the point of view of an explorer: assume nothing, expect anything. Its method was to build one corner according to an arbitrary given definition, or “signature”, and then to continue propagating corners until all unconnected edges joined with preexisting edges, thereby completing construction. As more and more corners were built, a prospective solid would gradually emerge. If there was no way to bridge all surface gaps with perfectly conforming corner constructions before faces began intersecting, construction would be abandoned. On the other hand, if all edges eventually joined before faces intersected, then the solid would have been empirically proven to exist. In Archimedean's early stages, rendering logic was only capable of displaying fully convex surfaces (adequate for the Platonic and Archimedean Solids). I noticed that it was able to mathematically construct some unfamiliar objects, but that it was unable to render them well. I was left wondering what this meant.

Reworking Archimedean's construction and rendering logic to be able to handle star polygons and intersecting faces enabled Archimedean to handle all known uniform polyhedra, but more importantly, it allowed me to get a better glimpse of those aforementioned unfamiliar objects. Seeing them verified my suspicions. Like the apprentice in Walt Disney's “The Sorcerer's Apprentice”, if Archimedean was not specifically instructed to stop building when faces started intersecting, it would continue to build and connect corners forever until all edges joined or it ran out of memory. For most of these unfamiliar signatures, thousands upon thousands of corners would be built, eventually resulting in an out-of-memory condition. For some signatures, however, an extra pass around the surface was all that was necessary for all edges to meet. “Kasparian Constructions” are the result of relaxing the condition that the faces of a polyhedron may traverse the surface only once in an attempt to join all edges.

Algorithm for Constructing any Uniform Polyhedron or Kasparian Construction

1: Joining faces edge to edge around a single point, construct a corner according to the given signature. Two faces with a common edge that are used (or reused) to construct the same corner are called a “facepair”. At this point, every facepair created has been used in the construction of one corner only.

2: Selecting a facepair that has been used in the construction of only one corner, construct a second corner around the other endpoint of that facepairs common edge. Reuse the faces of the facepair as well as any other existing faces if they are consistent with the given signature.

3: Add the newly-formed facepairs to the growing collection of facepairs.

4: Repeat steps 2 and 3 until all facepairs have been used/reused in the construction of two corners: one for each of the endpoints of the facepairs common edge.

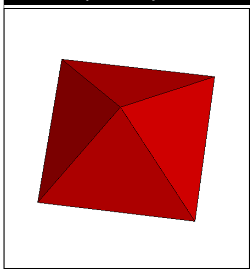
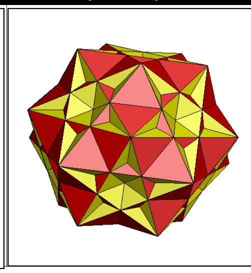
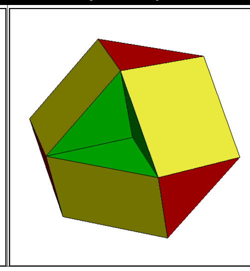
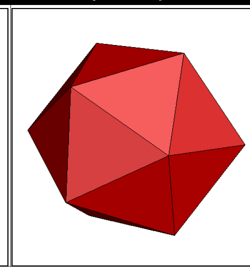
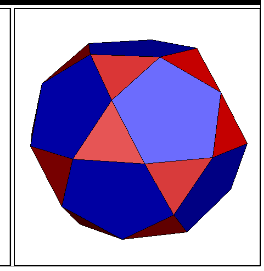
Figure 2: *Archimedean's Construction Algorithm.*

	Conditions							
Classification	convex polygons	one type of polygon	disallow double polygons	convex vertices	each vertex the same	faces do not intersect	faces traverse the vertex only once	faces traverse the surface only once
Platonic	✓	✓	✓	✓	✓	✓	✓	✓
Archimedean	✓	relaxed	✓	✓	✓	✓	✓	✓
Kepler	relaxed	✓	✓	✓	✓	relaxed	✓	✓
Poinsot	✓	✓	✓	✓	✓	relaxed	relaxed	✓
Non-regular Star Polyhedra	relaxed	relaxed	✓	relaxed	✓	relaxed	relaxed	✓
Johnson	✓	relaxed	✓	✓	relaxed	✓	✓	✓
Taylor	relaxed	relaxed	relaxed	relaxed	✓	relaxed	relaxed	relaxed
Kasparian	relaxed	relaxed	✓	relaxed	✓	relaxed	relaxed	relaxed

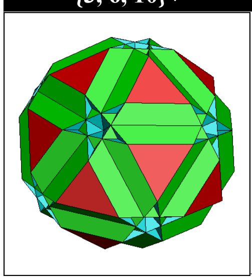
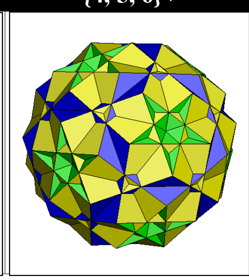
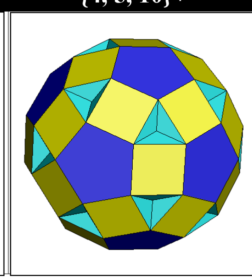
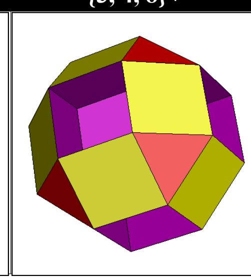
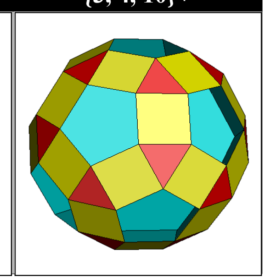
Figure 3: *Characteristics of various polyhedral classes and Kasparian Constructions.*

Kasparian Constructions

In Poinsot Solids and some of the star polyhedra, faces intersect either because they wind around the vertex twice before joining each other, or because the faces themselves are made from polygons with intersecting line segments. In Kasparian Constructions, faces intersect because they pass around the surface more than once before joining each other. As they begin their subsequent passage around the surface, the vertices, edges, and sometimes faces coincide in terms of their positions in coordinate space. When they coincide, it is because they belong to logically distinct corners, each of whose construction requires the same vertex, edge, or polygonal face. Consequently, these constructions, like Skilling's figure, are degenerate. Figure 4 describes the first ten Kasparian Constructions. If coinciding edges and faces are considered to be distinct from each other, then some edges are really multiple coinciding edges, each formed by two polygonal faces meeting. If, on the other hand, coinciding edges and faces are not considered to be distinct, then some edges are branches where more than two polygonal faces meet.

K1 **{3, 3, 4}+**	**K2** **{3, 4, 5}+**	**K3** **{3, 4, 6}+**	**K4** **{3, 5, 5}+**	**K5** **{3, 5, 10}+**
				

K1 resembles an Octahedron with an inner framework of three squares.
K2 contains 5 cubes plus an inner framework of pentagons and a shell with pentagonal holes that is made of intersecting triangles. It shares the same vertices as the Great Ditrigonal Icosidodecahedron, but connects them differently.
K3 shares the same vertices as the Cuboctahedron, but connects them differently.
K4 can be decomposed into two known solids, the Icosahedron and the Great Dodecahedron.
K5 resembles the Icosidodecahedron with an inner framework of 6 decagons.

K6 **{3, 6, 10}+**	**K7** **{4, 5, 6}+**	**K8** **{4, 5, 10}+**	**K9** **{3, 4, 8}+**	**K10** **{3, 4, 10}+**
				

K6 - K10 do not contain within themselves any other convex polyhedra. They do share the same vertices as other known polyhedra.

Figure 4: *Descriptions of Kasparian Constructions 1 - 10.*

From this second viewpoint, two of these constructions resemble Grünbaum's (1-2-3)-Complexes [2]. These are constructed by adding polygonal faces to known polyhedra in order to create solids where three faces share an edge. Grünbaum discusses the octahedron with an inner framework of intersecting squares, and the icosidodecahedron with an inner framework of intersecting decagons. These are identical to K1 and K5 if coinciding faces in K1 and K5 are not considered to be distinct, the only difference being the method of construction. My method focuses on corners, whereas Grünbaum's focuses on edges.

Many Kasparian Constructions are identical in their 3-D realizations, yet distinct in their derivations, from polyhedra discussed by Taylor [3] that utilize double polygons. A double polygon has an even number of edges winding around its center twice. Taylor's Triquasitruncated Icosidodecahedron uses a double triangle in its vertex figure, but is otherwise indistinguishable from K10, which uses a normal triangle. The same is true for the Triquasitruncated Octahedron and K3, as well as others. In general, double polygons in Taylor's polyhedra correspond to coinciding faces in Kasparian Constructions.

The signature of a Kasparian Construction lists the types of polygons used to construct each corner, separated by a comma and surrounded by curly brackets. A plus sign is appended to signify that the surface is wrapped more than once. For example, the signature of K7 is {4,5,6}+ indicating that a square, a pentagon, and a hexagon meet at each vertex as the solid is constructed, and that the faces wrap around the center of the solid more than once. It should be noted that coinciding faces and edges ultimately result in more than the three signature faces sharing each vertex. This is because two differently oriented corners are built at each vertex. Visual inspection of K7 reveals that each of its vertices belongs to two squares, two hexagons, and one pentagon (two coinciding pentagons). If one doesn't consider coinciding edges and faces to be distinct, then K7 has 2 types of edges: some where three faces meet (square, pentagon, and hexagon) and others where 2 faces meet (square and hexagon).

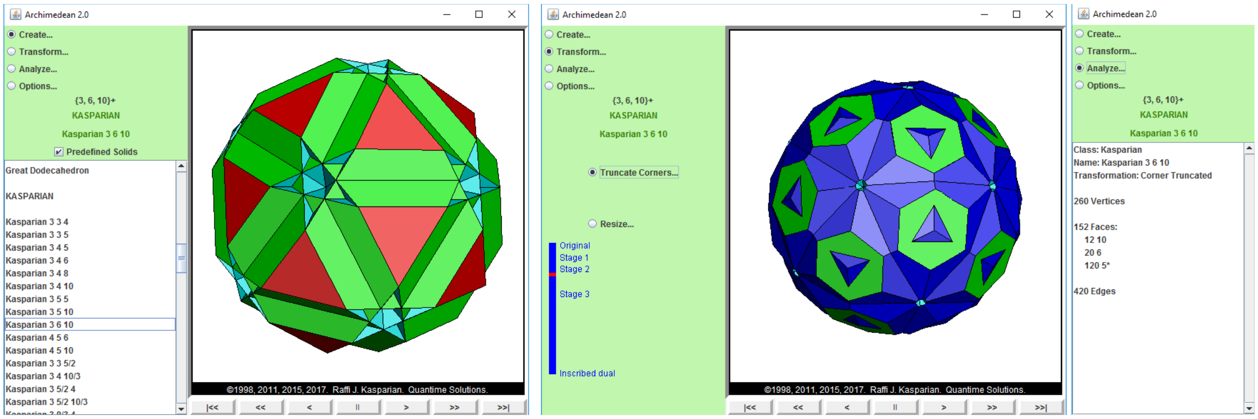


Figure 5: *Screenshots of Archimedean 2.0 interface showing K6 and a corner-truncated K6.*

Archimedean's interface delivers a "hands-on" feel of object manipulation, as well as providing an accurate and intuitive way of transforming and analyzing polyhedral solids and constructions. For example, Archimedean demonstrates step-by-step construction as well as edge and corner truncation of all the uniform polyhedra and Kasparian Constructions. Duals exist and may be displayed for all constructions, except for a few of the Kasparian Constructions where faces pass through the origin.

Conclusion

Through the centuries, new polyhedral classifications were generally found when necessary and sufficient conditions were incrementally relaxed. In particular, just as Poinsot Solids arose from allowing faces to wind around a vertex more than once, Kasparian Constructions arose from allowing faces to traverse the surface more than once. Carlo H. Séquin, Professor Emeritus in the Graduate School, UC Berkeley, who has kindly considered their nature, has described them as "symmetrical assemblies of regular n-gons sharing edges and vertices in a 'regular' way, so that at every vertex the same configuration of edges and polygon corners results." Kasparian Constructions seem a natural extension to the ever growing application and exploration of the field of polyhedra, and will hopefully be of practical, aesthetic, and theoretical interest to the mathematical community. Recently, using Archimedean to systematically explore other possible vertex signatures resulted in the discovery of several additional constructions. All of these constructions may be viewed and further explored in Archimedean 2.0, which may be found at `http://www.quantimegroup.com/solutions/pages/Archimedean2.0/KasparianConstructions.html`.

References

[1] P. R. Cromwell. *Polyhedra*. Cambridge University Press. 1997.
[2] B. Grünbaum. (1-2-3)-Complexes. Geombinatorics 13(2003), 65 - 72.
[3] P. Taylor. *The Simpler? Polyhedra*. Nattygrafix. 1999.