Invertible Infinity: A Toroidal Fashion Statement

Ellie Baker
Lexington, MA 02421, USA
ellie.baker@post.harvard.edu

Charles Wampler
Applied & Computational Math. & Statistics
University of Notre Dame
Notre Dame, IN 46556, USA
cwample1@nd.edu

Abstract

We present the design of a uniquely constructed reversible "infinity scarf"—a specially made cloth torus such that its shape is invariant under inversion *and* it folds flat into a six-layer equilateral triangle. Since the meridians and longitudes of a torus swap places under inversion, one might think the shape invariance property dictates construction from a square piece of fabric (with opposite edges sewn together). However, although inversion invariance can be achieved with a square construction, the perfect equilateral triangle folding cannot. The triangle folding is a special case of what we refer to as "ribbon folding" and we show that our scarf and one made from a square are the only toroidal forms made from flexible but inextensible fabric that are invariant under inversion and fold in a planar ribbon loop. We present several fabric layouts that can be used to produce the scarf, along with sewing instructions, and show that, among all such layouts, the seam length of the hexagon is the shortest possible.

Introduction

For artists, craftsmen, and designers, fun mathematical puzzles abound. Here's one in the realm of fashion design.

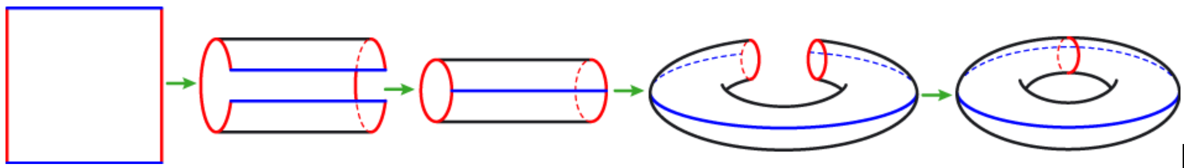
Suppose you want to create a reversible "infinity scarf"—a traveler's ideal, versatile item of clothing—two outfits in one. For mathematicians among you who don't know what an infinity scarf is, the classic design is a cloth torus whose hole is large enough to fit over the head so that it can be looped around the neck one or more times. For fashion enthusiasts among you who don't know what a torus is, it's basically a donut shape—and if you don't immediately see a relationship between a donut and your favorite infinity scarf, imagine stuffing it like a pillow (and perhaps shrinking it a bit) and you should get the idea. Having never before encountered a truly reversible infinity scarf[1], we wondered why not. Is it possible to design and create one that is an attractive, functional garment?

Constructing a Torus from Fabric and the Inversion Problem

The mathematical notion of a *torus* as a stretchy, shrinkable square, whose opposite edges are understood to be glued together (Figure 1) suggests a possible sewing construction for an infinity scarf: a square piece of fabric whose opposite edges are *stitched* together. Because real fabric, even very stretchy real fabric, doesn't have the unlimited stretchiness and shrinkability presumed by topologists, a practical seamstress might think to start with a long, thin rectangle instead of a square, a tactic that essentially

[1] Our Internet searches on "reversible infinity scarves" produced many results, but all turned out to be two-sided bands whose different sides were both potentially visible when worn. Our definition of a "reversible infinity scarf" is different, and is limited to cloth tori with an interior surface that is different from the exterior surface and that can be seen only when the torus is inverted, i.e., turned inside out.

provides a little advance stretching in one direction, in order to construct a scarf that has an appropriate geometry and shape for a typical infinity scarf. An inspection of commercially made toroidal infinity scarves reveals that they are generally made using this exact construction.



[5]

Figure 1: *A topological torus: a stretchy shrinkable square whose opposite edges are understood to be connected.*

Although it may not be intuitively obvious, a savvy seamstress will know that, just like a pillow cover, a toroidal cloth form can be inverted[2] (i.e., turned inside out) through a small hole in the fabric or along a seam. And, if the torus is made from very thin, silky fabric, this inversion hole can be quite small. However, a difficulty arises with a scarf sewn from a rectangle when the scarf is inverted because the meridians and longitudes of a torus swap places under inversion.[3] Figure 2 illustrates this surprising (at least to non-topologists) phenomenon. Due to the swap, a toroidal scarf constructed from a long, skinny rectangle will dramatically change geometry and shape when inverted. Depending on the dimensions of the rectangle, the inverted scarf may no longer even be wearable. Seamstresses, who typically sew an object inside out and then invert it to hide the seams on the interior, must account for this in their construction methods [3]. Since our fashion design goal is to create a reversible scarf that is identical in shape and size when turned inside out, this seems like a potentially serious design obstacle! Our plan is to sew it with reversible fabric and we want the only change on inversion to be the change in visible surface pattern. Does this mean a square construction is our only option for achieving this? We'll return to this question in a moment, but first let's consider another important design specification.

Figure 2: *Stages of a punctured, stretchy, shrinkable torus being turned inside out. These are still images from an online animated gif by Surot [2] showing how the meridians (circles going through the torus' hole) become longitudes (circles going around the torus' hole) on inversion, and vice versa.*

Further Complications: The Möbius Twist

In order to facilitate a flatter (and more flattering) drape on the human figure, infinity scarves are often constructed with a Möbius half twist. We can create a Möbius-like toroidal scarf from a rectangle by sewing together the two long edges first, and then giving the resulting cylinder a half twist before sewing together the short edges (the cylinder ends). This process will produce a scarf that looks something like the leftmost drawing in Figure 3. If the larger dimension of the rectangle is long enough, the resulting scarf will loop around the neck one or more times, and the Möbius shape creates a more functional and aesthetically pleasing garment.

---

2 For the purposes of this write-up, "invert" implies the same transformation as "evert" in the topological literature.

3 See this presented as a puzzle in The New York Times Numberplay blog [3]. We encourage readers who want to gain tactile intuition for this strange phenomenon to try a quick experiment with sewn cloth, or cut paper and tape. With cloth, sew together the opposite edges of the rectangle, leave a small hole in one seam, and try out the inversion. With paper and tape, make two identical rectangles and check out the difference depending on which set of opposing edges you join first (the join order determines which edge becomes a meridian circle and which a longitude circle of the torus).

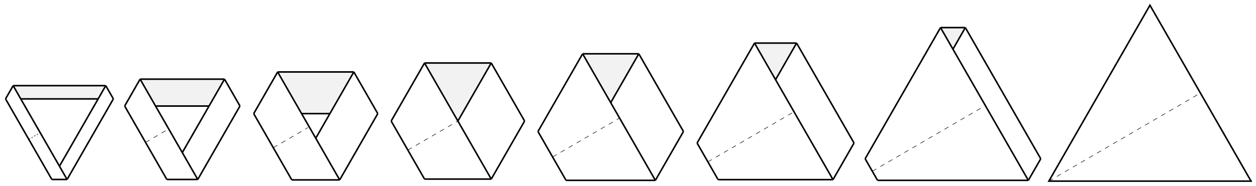


Figure 3: *Infinity scarves are often constructed with a Möbius half-twist to facilitate a nice drape on a human figure. If we preserve the triangular folding of the Möbius twist while gradually increasing only the scarf width (i.e., the shorter dimension of the original rectangle), the process reaches a limit when the scarf becomes a six-layer equilateral triangle (right). What shape/dimensions is the resulting layout now?*

Unfortunately, due to the swapping of meridians and longitudes, the inverted version of the leftmost scarf depicted in Figure 3, although still toroidal, winds up looking more like a windsock than a scarf—neckwear perhaps suitable for a giraffe, but not particularly appropriate for a human being.

The left to right image progression in Figure 3 illustrates what happens if we explore counteracting this problem on a toroidal Möbius scarf by gradually increasing the shorter dimension of the original rectangle, while keeping the longer dimension unchanged. This progression need not end at the form with a hexagonal perimeter (shown fourth from left), where the visible triangular hole in the center shrinks to a point. It can continue until the scarf becomes a six-layer deep equilateral triangle (shown at far right). What is the rectangular layout of the resulting object now? Have we arrived at a square?

Interestingly, the answer is no—we've actually gone beyond it[4]—and the object we've produced has the curious property of being shape-invariant on inversion without having a square layout. Furthermore, it preserves some excellent properties from a fashion perspective—namely, it still fits over the head (the size of the hole, although hidden in the folds, is unchanged[5]) and it retains its nice Möbius drape. In fact, as we shall see in the next section, other than a square-based design, it's the only "ribbon-foldable" toroidal scarf that is shape-invariant on inversion. We can also create paper and cloth models that permit tactile exploration of our curious new object, allowing us to gain a great deal of intuition about it while verifying some of the scarf's interesting properties.

Ribbon-Foldable Tori and Their Inversions

It is advantageous for a scarf to be able to be folded neatly onto a plane: for the neat-freak, this makes it easier to store, but more importantly, it enhances the attractiveness of how it drapes on the body. One might devise complicated, origami-like, folding patterns, but we will limit our consideration to a simpler approach that we call *ribbon folding.* We assume that the toroidal scarf is constructed starting with a cylindrical tube, such as the middle image in Figure 1, which is then closed up by bringing the two open ends together. Clearly the tube can be collapsed onto a plane to take a rectangular shape two layers deep. One may then make a series of straight folds, each spanning from one long edge to the other, until the open ends are brought into alignment. Figure 4 shows a cylindrical tube being collapsed to a rectangle and then folded once. Figure 5 shows several possibilities for folding the rectangle into a closed loop. The long edges of the rectangle may be considered as the two edges of a ribbon, and we are considering ways of folding this ribbon onto a plane until its ends meet, forming a planar ribbon loop. (There exists a literature on closed ribbons: for example, see Dennis and Hannay [4].) We define a *ribbon-foldable torus* as any (inextensible, infinitely flexible) cloth torus that can be manipulated into such a double-layer planar ribbon loop.

[4] The original width has now become slightly longer than the original height.

[5] In Figure 3, the rectangle length is held constant while the width changes, so the size of the head opening is held constant.

Planar ribbon loops can be categorized as even or odd according to the number of folds. When closing up an even loop, each edge of the unclosed rectangle mates back to itself, resulting in a two-sided ribbon. In Figure 4, this means point A mates to B, and D to C, so each of $\overline{AB}$ and $\overline{CD}$ closes up into its own loop. For an odd ribbon loop, each edge mates to the other, i.e., point A to C and point D to B, thereby forming a Möbius strip with only one closed edge, $\overline{ABDCA}$. In Figure 5, we have labeled each fold as contributing either $+1/2$ or $-1/2$ twist to the ribbon loop.

To better understand a *ribbon-folded torus*, we reverse the construction procedure described above: cut across the torus to get a cylinder of cloth as in the top image of Figure 4, cut along longitudinal line AB, and unroll the cloth into a rectangle. We then place that rectangle in the plane with AB vertical, as in Figure 6. Line DC is parallel to AB and bisects the rectangle. To convey how the edges match up when the torus is sewn closed, we may tile the plane with copies of the basic rectangle. Repeating the rectangle horizontally shows how the edges meet when we wrap the cloth into a cylinder. Similarly, when adding a second row of rectangles above the initial row, we align them according to how the ends are sewn together; that is, we need to place the next copy of A on top of the point along line BCB to which it is sewn when closing up the torus. When we limit ourselves to tori closed up by ribbon-folding the flattened cylinder, point A must match either to point B (for even loops) or point C (for odd loops), resulting in the two different grids shown in Figure 6, which we will refer to as even and odd tilings, respectively.

The tilings help us see what happens when the torus is inverted. First, we need to indicate on the diagram the meridians and longitudes. These are closed loops of minimal length going once around the torus, either around its "tunnel" for a meridian or its "hole" for a longitude. Minimal length curves are straight lines in the tiling diagram. For both even and odd tilings, meridians run horizontally. For even tilings, longitudes run vertically, and thus the meridians and longitudes run parallel to the sides of the basic rectangle. In contrast, the longitudes for odd tilings run on a slant, advancing one half-width of the basic rectangle while advancing one unit in height. Drawing one meridian and one longitude (and their repetitions) in the tiling produces the parallelograms overlaid on the odd tiling in Figure 6.

Inverting the torus does not change the basic geometry of its plane tiling, but the lines must be reinterpreted. Meridians become longitudes and vice versa. For the even tiling, little appears to change: we just rotate the diagram 90º. This reflects the fact that every cloth torus ribbon-folded with zero twist still ribbon folds after inversion, although, unless the tile is square, the result has different dimensions.

The situation is quite different for odd tilings. If the inversion is ribbon foldable, then we must be able to overlay a new odd tiling that uses the former longitudes as the new meridians. The upper-left image in Figure 7 shows what typically happens when the torus for an odd ribbon loop is inverted: the new basic rectangle A'B'EF built on new meridian A'F fails to match the geometry of the torus, because C' does not coincide with C. This means that the inversion is *not* odd-ribbon-foldable. (Not even-ribbon-foldable either, for that matter.)

However, all is not lost. One can determine what proportions of the parallelogram produces odd tilings in both inversions. Rather than answer that question here (the reader may wish to ponder it), we restrict our attention to tori that are shape invariant under inversion. Such a torus must have meridians and longitudes of equal length, i.e., the basic parallelogram must be a rhombus. One may confirm that only the 60º rhombus in the upper-right image of Figure 7 produces an odd tiling in both directions. For clearer viewing, the bottom image repeats the pattern to show three tiles in each direction.

So far, we have shown that the 60º rhombus gives the only odd tiling that can possibly self-invert. It is beyond our scope here to rigorously prove that this necessary condition is also sufficient, but at any rate, it is more satisfying to actually construct the torus and physically confirm this fact. And what scarf results from this? As we will show shortly, the rightmost image in Figure 3 is the result of ribbon folding the 60º design using a $+1/2$-twist. The twist is not clear in that image, but it becomes clear by observing that twist is preserved as we progress left to right across the sequence in that figure. It is less obvious what will happen to the twist upon inversion, but it turns out that the inversion also has a $+1/2$ twist (it does not flip to $-1/2$) so the torus is perfectly invariant under inversion.

In summary, we have established that there are just two ribbon-foldable torus designs that are shape invariant under inversion: the square design for even loops and the 60º rhombus for Möbius-twisted loops. Next, we demonstrate our claim that the rightmost image in Figure 3 corresponds exactly to the 60º design.

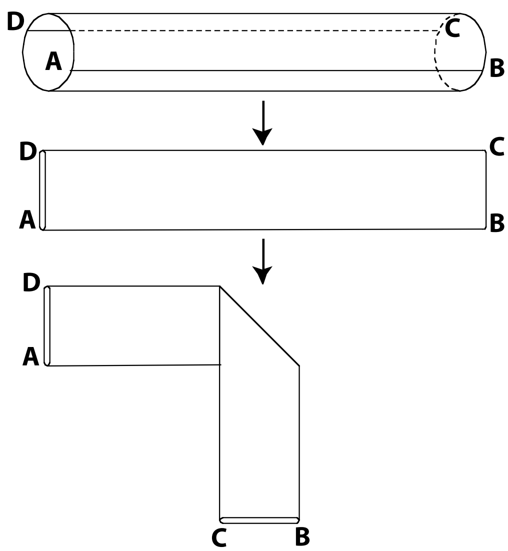


Figure 4: *Circular cloth cylinder collapsed to a planar rectangle, then folded once.*

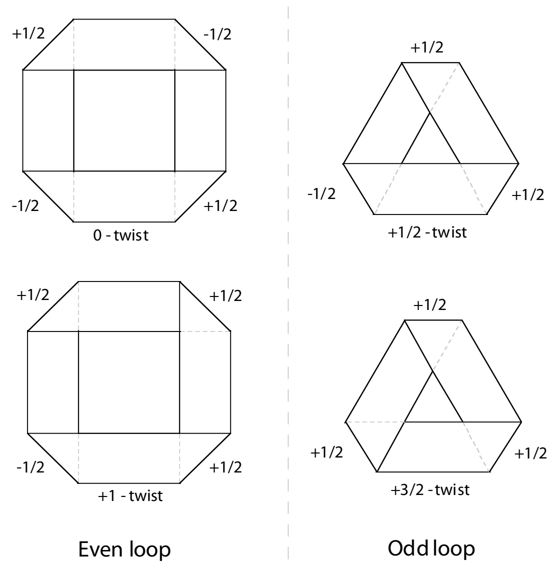


Figure 5: *Even and odd planar ribbon loops, with twist indicated.*

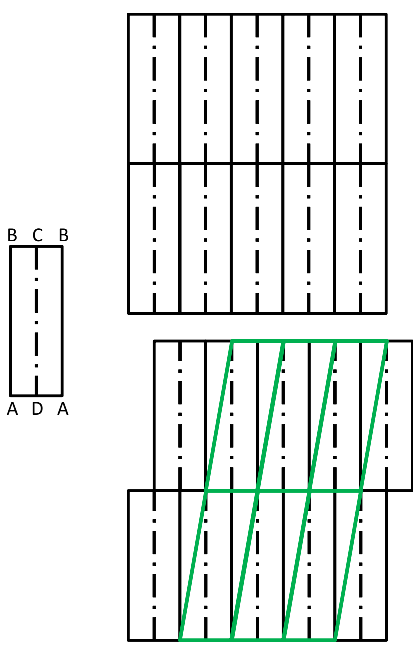


Figure 6: *Tilings compatible with even and odd planar ribbon loops.*

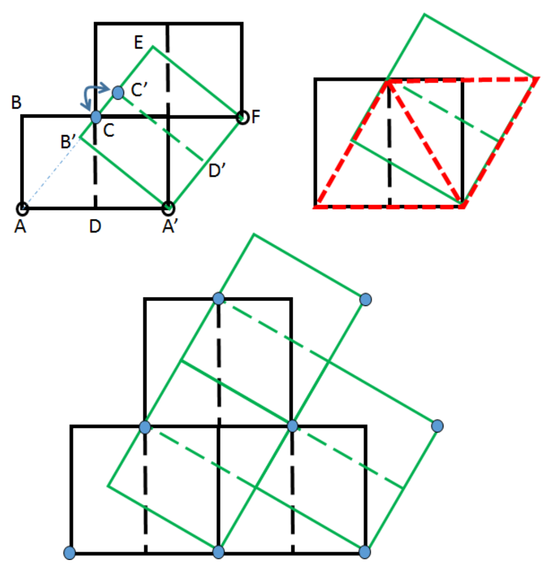


Figure 7: *A general odd ribbon loop does not invert to an odd ribbon loop. Only a design based on equilateral triangles inverts to the exact same odd ribbon loop.*

Layout Options

To visualize the rectangular layout needed to create the six-layer triangular scarf shown at the far right of Figure 3, we can deconstruct it with a thought experiment as described in Figure 8. Reversing the steps shown there, folding and sewing instead of unfolding and cutting, provides sewing instructions to produce the scarf from the layout. Inspecting the dotted fold lines of the resulting rectangular layout (Figure 8 right), we can now clearly see all six equilateral triangles, although two are sliced in half by a seam along a perpendicular. The rectangle's aspect ratio of $\frac{\sqrt{3}}{2}$ is clear from the equilateral triangle.

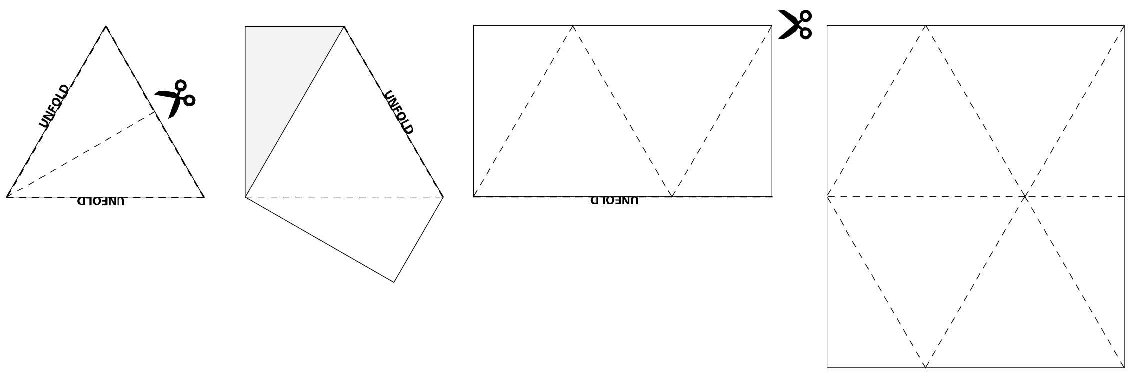


Figure 8: *Thought experiment to deconstruct the Möbius-folded six-layer equilateral triangle on the far right of* Figure 3 *(steps shown left to right): 1. Cut two layers deep along the dotted meridian line and unfold the two resulting triangular flaps. 2. Unfold the trapezoidal flap. 3. Cut the resulting cylinder open along the top edge, and unfold it open.*

With a little experimentation moving the seam lines around while retaining the same area, we can confirm that a hexagon and a 60° rhombus, as shown in Figure 9, are also valid layouts. The edge labels in Figure 9 indicate how edges connect when sewn. The largest (black) dashed lines represent a set of meridian and longitude lines on the resulting torus, running 60° apart. The remaining dashed lines are fold markings that can be used to produce paper models of the six-layer equilateral triangle by folding first red (small dash), then green (medium dash), and finally blue (larger dash). If enlarged and printed on white paper, each layout can be folded into a paper model with either the colored or white surface showing, demonstrating the inversion invariance. In general, the sewing instructions involve simply sewing together the opposite edges in any order—except for the rectangle, where the matching labels on its long edges are not directly opposite one another. Sewing together the rectangle's short edges first and giving the resulting cylinder a half twist before sewing together the long edges, produces the correct alignment.

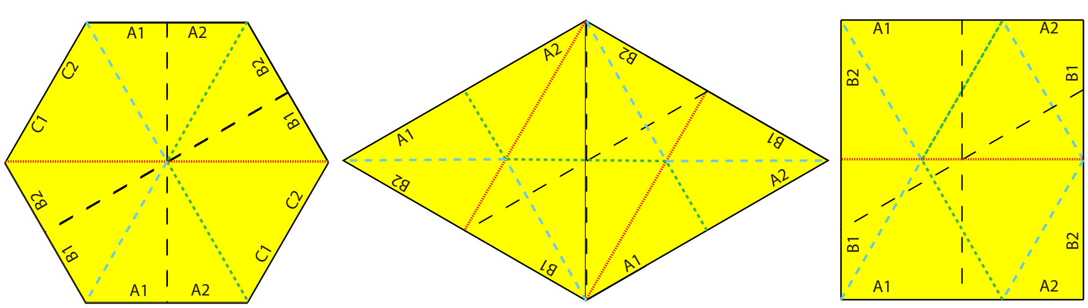


Figure 9: *Three possible sewing layouts and/or paper models. (Note: for sewing, add seam allowances.)*

Seam Length

An interesting distinction between the layouts in Figure 9 is that they have different total seam lengths, with the hexagon notably shortest. The seam length is half the layout's total edge length. Using an equilateral triangle side of 1, the total edge lengths are: Hexagon = **6**; Rectangle $= 3 + 2\sqrt{3} \approx$ **6.5**; Rhombus $= 4\sqrt{3} \approx$ **6.9.** But is there anything shorter than the hexagon among all possible layouts?

The "Honeycomb Conjecture," proven in 1999 by Hale, states that "any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling."[1]. Since any valid scarf layout tiles the plane with tiles of equal area (Figure 10), there is no layout with a seam length shorter than the hexagon. For this reason, we consider a hexagon the quintessential layout for our scarf.

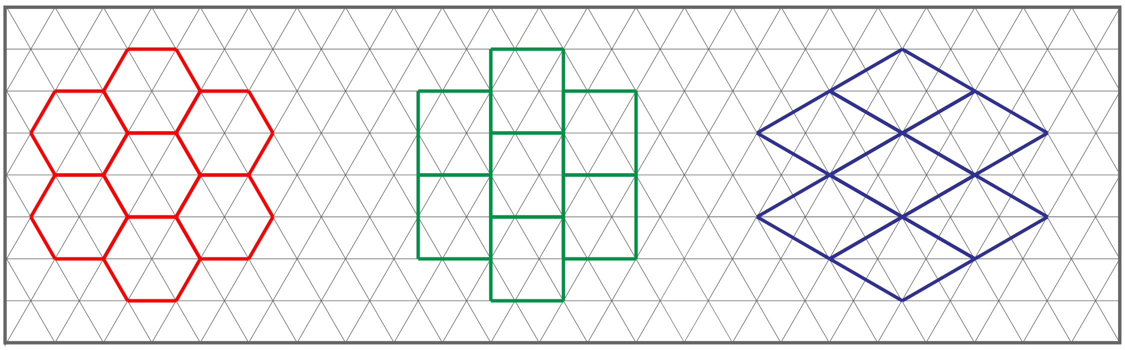


Figure 10: *Any valid scarf layout has area equal to six equilateral triangles and will tile the plane with tiles adjoined at the sewing seams.*

Fabric Design and Sewing Notes

The specialness of the honeycomb tiling, and its relationship to our scarf geometry, suggests another possible design element. Fabric printed with a P6 symmetry pattern [5] nicely echoes the hexagonal structure of the scarf and, if printed at a suitable scale, also facilitates exact matching of the pattern at the seams, regardless of the layout used. Two apps, iOrnament (pixel-based) and Kaleidopaint (spline-based), enable easy creation of wallpaper group patterns, and both output a tile image that can be input directly to fabric printing venues like Spoonflower. Spoonflower will not print reversible fabric, but it's possible to create your own using two fabric layers attached back to back. Figure 11 shows a rendition of the scarf with P6M patterned fabric made this way. (Mathematically OCD designers will be dismayed to learn that Spoonflower's printing process can shrink the fabric inconsistently along the warp vs. the weft, slightly—but manageably—throwing off the precise P6 symmetry. This is more noticeable on silk than on polyester crepe de chine.)

From a sewing perspective, seam length is not the only thing we might want to optimize. Ease of construction is also an issue, and our experience is that seams intersecting at right angles are easier to sew, which may make the rectangular layout preferable. It is also usually the layout with the least trim waste. On the other hand, in addition to the benefit of shorter seams, one might argue that the hexagon better elucidates the scarf's mathematical structure and that its seams add a further decorative and more mathematically meaningful element to the design.

Another potential sewing design consideration is placement of the layout with respect to the fabric warp. Clothing designers will sometimes cut fabric "on the bias" in order to line up a seam at a 45 degree angle from the warp and weft, which improves drape and stretch along that dimension of the garment. Such details are the purview of sophisticated clothing designers, who may wish to align the scarf's toroidal axes so they are both at the same 30° angle from the fabric warp—to subtly enhance the wearer's sense of inversion invariance by giving the scarf neck hole the exact same degree of stretch on both sides.

Figure 11: *A rendition of the scarf,* Ellie Baker's "*Invertible Infinity," shown at the Joint Mathematics Meetings 2017 exhibit of mathematical art. The "filmstrip" at the bottom shows the inversion progressing from one equilateral triangle folding to the other. The equilateral triangle side length of this scarf is about 21 inches. It was constructed using French seams.*

Conclusion

In addition to being a functional and fashionable garment, the scarf is itself a puzzle. One puzzle a wearer might want to solve first is figuring out how to fold it into the six-layer equilateral triangle. Doing so after inverting—and prior to donning—is useful because it helps "find" the nice Möbius drape. It's also a convenient way to store or pack it. Here are a few other scarf-related conundrums a wearer might enjoy pondering: With its one-half ribbon twist, the hexagonal scarf is invariant under inversion—would it also be inversion invariant if constructed to have a 3/2 ribbon twist? How about a square scarf with one full ribbon twist? Ribbon folding is only compatible with integer half-twists—is there some more general planar folding scheme that allows other twist values?

Our scarf, with its novel design, dimensions, and structure, has the confluence of features needed in a reversible infinity. It is the *only* Möbius ribbon-foldable toroidal scarf that is shape-invariant under inversion. Its property of inverting to exactly the same dimensions is not just a nice math result, but makes the scarf doubly useful in a fun convergence of the theoretical and the practical.

References

[1] Thomas C. Hales. "The Honeycomb Conjecture", *Discrete and Computational Geometry*. 25(1): 1—22. 2001, arXiv:math/9906042 [math.MG].

[2] Surot. Public domain animated gif at: `https://commons.wikimedia.org/wiki/File:Inside-out_torus_(animated,_small).gif` (as of Jan. 7, 2017).

[3] Gary Antonick. "Ellie Baker and the Seamstress's Conundrum", *The New York Times Numberplay Blog,* August 24, 2015.

[4] M.R. Dennis and J.H. Hannay. "Geometry of Calugareanu's theorem," *Proc. R. Soc.* A (2005) 461, 3245–3254

[5] Ellie Baker and Susan Goldstine. *Crafting Conundrums: Puzzles and Patterns for the Bead Crochet Artist*. AK Peters/CRC Press. 2014.