Inversive Diversions and Diversive Inversions

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Abstract

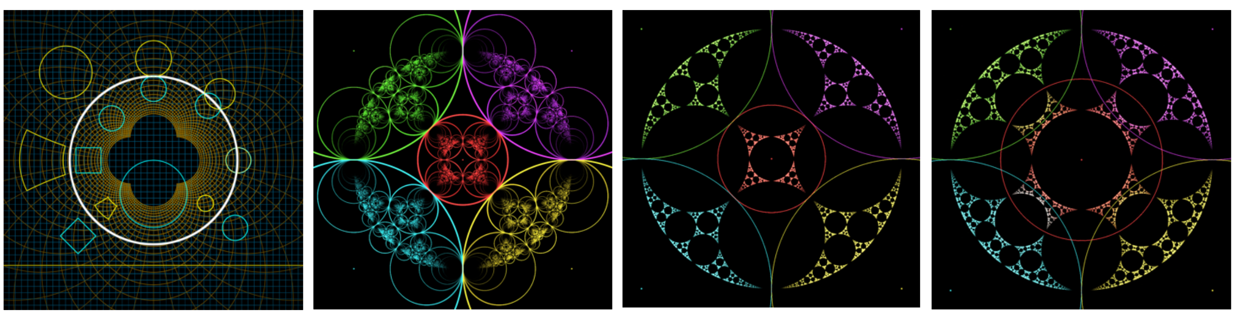
In this paper, we first briefly review aspects of inversive geometry and inversive fractals. Motivated by a desire to expand our geometric and artistic toolkit, we then introduce *mixed-restriction limit sets* as a new technique for use with iterated function systems and groups of circle inversions to create previously undiscovered 2D inversive fractals. We also apply this technique to diverse shape-based inversions that are closely related to circle inversion. Finally, we explore extending mixed-restriction limit sets and shape-based inversions into 3D to generate 3D inversive fractals.

Circle Inversions and Inversive Fractals

Given a circle C_0 in the 2D Cartesian plane with center at the origin (0,0) and radius r, the equation for the circle inversion transformation I_{Co} of a point (x,y) relative to the circle to yield a new point (u,v) is:

$$(u, v) = I_{Co,}(x, y) = \left(\frac{xr^2}{x^2+y^2}, \frac{yr^2}{x^2+y^2}\right) \text{ with additional definitions that } I_{Co}(0,0) = \infty \text{ and } I_{Co}(\infty) = (0,0)$$

For a circle $C_{a,b}$ with center at (a,b) the inversion transformation $I_{Ca,b}$ can be composed from circle inversion centered at the origin and translations: $I_{Ca,b} = T_{a,b} \bullet I_{Co} \bullet T_{-a,-b}$. Figure 1a shows how inversion transforms a square grid and various circles and squares placed at different positions relative to the circle of inversion.



1a) 1b) 1c) 1d)

Figure 1: *a) Square grid (bounded by figure border) and shapes transformed by circle inversion relative to the large circle, b) iterative inversion of five tangent circles, c) fractal limit set of same circle inversion configuration as in 1b, d) same as 1c but with larger inner circle intersecting outer circles with angle π/3*

Circle inversion transformations have many interesting properties:

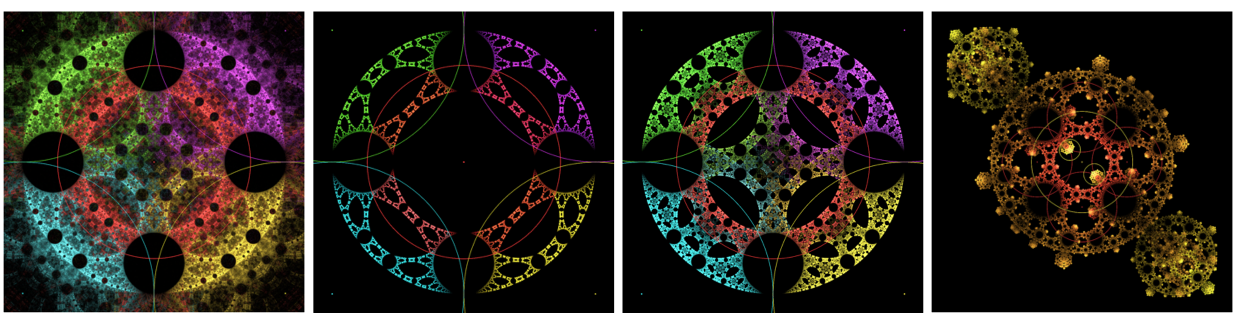
- Any point p on the circle of inversion is unchanged by inversion, $I_C(p) = p$
- I_C transforms all points outside of the circle to inside the circle, and vice versa
- I_C is a contraction for all points outside the circle, and an expansion for all points inside the circle
- I_C is an involution at every point: $I_C(I_C(p)) = p$
- I_C is an anticonformal mapping (preserves local angles but reverses orientation)
- I_C transforms circles to either circles or lines, and lines to either lines or circles

With certain configurations of tangent circles, iterative inversion of each circle boundary relative to the initial circles appears to converge towards complex structures, as illustrated in Figure 1b. To examine this further we recast the same group of circle inversions in Figure 1b as an iterated function system (IFS) [1]. Starting with a random point, the IFS iteratively transforms the point by picking randomly amongst the inversions to apply at each iteration and accumulating the resulting points. After discarding initial points, this forms an approximation of the limit set of the group of inversions, which is a compact but complex fractal (see Figure 1c) as first shown for groups of circle inversions by Mandelbrot [2].

Restricted Limit Set Fractals and Mixed-Restriction Limit Set Fractals

Now consider a group of circle inversions where some pairs of circles are intersecting rather than tangent. If the angles of all the circle intersections are submultiples of π such that $\theta = \pi / n$ where n is a positive integer (kaleidoscopic angles [3]), then the limit set remains compact [4] (see Figure 1d). But for non-kaleidoscopic angles the limit set will not be compact and can extend over the entire plane (see Figure 2a). However, Clancy et al introduced *restricted limit sets* [5], which remain compact for any intersection angle. To construct a restricted limit set, start with a collection of circles $C_1,\ldots C_j$ and corresponding inversions $I_{C1},\ldots I_{Cj}$ as the group of transforms for an iterated function system. Then at each iteration step n, if the resulting transformed point P_n is inside circle C_k then I_{Ck} is restricted from being used as the transform for the next iteration n+1. This ensures there will be no expansive inversions, and thus that the restricted limit set that the IFS approximates is contained within the union of circles (see Figure 2b).

We now propose *mixed-restriction limit sets* as a simple but powerful modification to restricted limit sets that can generate diverse new fractals. In this construction, each inversive transform I_{Ck} can either be unrestricted (selectable in each IFS iteration to transform any point) or restricted (can only be selected as a transform for points outside the circle C_k). Figures 2a-c shows a comparison of different restrictions for a configuration of four outer tangent circles surrounding a fifth central circle that intersects the other four at a non-kaleidoscopic angle. In Figure 2a there are no restrictions, and the resulting limit set is noncompact. In Figure 2b all inversions are restricted, yielding a compact fractal limit set. In Figure 2c the outer circles are unrestricted but the inner circle is restricted, resulting in a different fractal limit set. In this example the limit set is still contained within the union of circles, but since mixed-restriction limit sets allow expansions this is not always the case (unlike restricted limit sets). Figure 2d shows an example of this, a configuration of seven circle inversions, all of which are restricted except for the centered circle which is unrestricted. Here the fractal limit set extends beyond the union of circles, yet it is still discrete and compact.



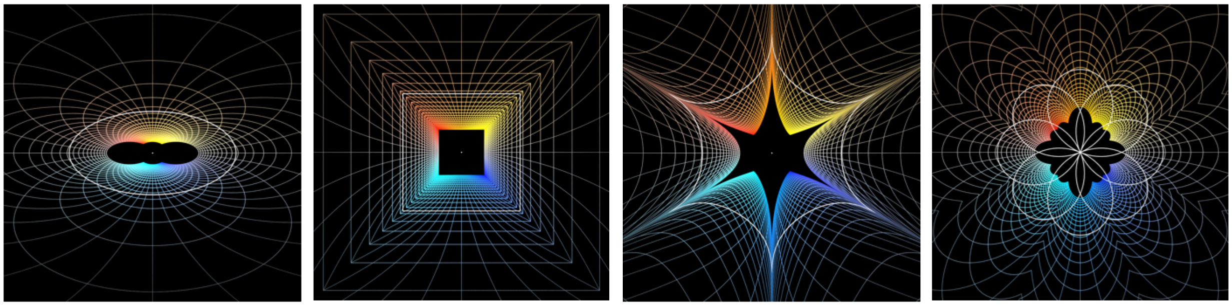
2a) 2b) 2c) 2d)

Figure 2a-c: *Comparing circle inversion fractals identical except for different restrictions, a) unrestricted, b) all five circle inversions restricted, c) only center circle inversion restricted. Circles of inversion are also shown. For each accumulated point, color indicates the last inversion applied.*
Figure 2d: *different group of seven circle inversions, all restricted except for center circle*

Generalizing Inversion to Other 2D Shapes

Gdaweic [6] introduced a generalization of inversion in a circle to inversion in *star-shaped sets* with circles, ellipses, and regular polygons all being special cases. For our purposes we propose a simpler subset of star-shaped sets that we will call *centered star shapes*, defined as any contiguous 2D shape S with boundary B and centroid O in which for all points M on B the line segment OM is contained entirely inside S (or on B). We can then define I_S, the shape inversion transformation for any point P = (x,y) with respect to S, as identical to the equation for circle inversion except that the circle radius r is replaced with the distance d from the centroid O to the intersection of the boundary B with a ray cast from O and passing through P:

$$(u, v) = I_S(x, y) = \left(\frac{xd^2}{x^2 + y^2}, \frac{yd^2}{x^2 + y^2}\right)$$

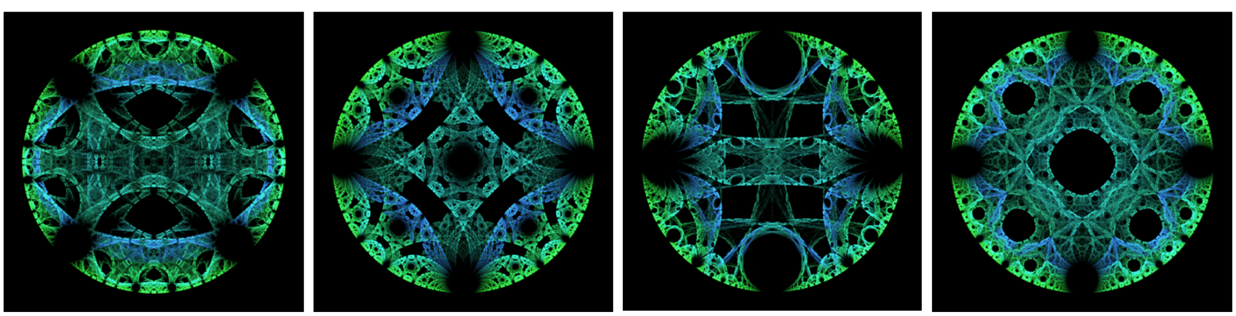


3a) 3b) 3c) 3d)

Figure 3: *Square grid (bounded by figure border) transformed by inversion in a) ellipse, b) square, c) supershape, d) rhodonea. The shape each inversion is based on is also shown as a thicker line.*

Figure 3 illustrates how a square grid is transformed by inversion in various centered star shapes. In addition to previously reported inversion in ellipses [7] (Figure 3a) and polygons [6] (Figure 3b) we have explored inversion in other shapes, including supershapes [8] (Figure 3c), and rhodonea curves [9] (Figure 3d). For shapes that are self-intersecting (such as some supershapes and rhodonea, depending on parameters) we convert them to centered star shapes by considering only the outermost shell relative to the centroid as the boundary of the shape (see Figure 3d). These inversions preserve many properties of circle inversion, for example any point on the shape boundary is unchanged, all points outside the shape are transformed to inside the shape and vice versa, and they are involutions so $I_S(I_S(p)) = p$.

Shape Inversion Fractals



4a) 4b) 4c) 4d)

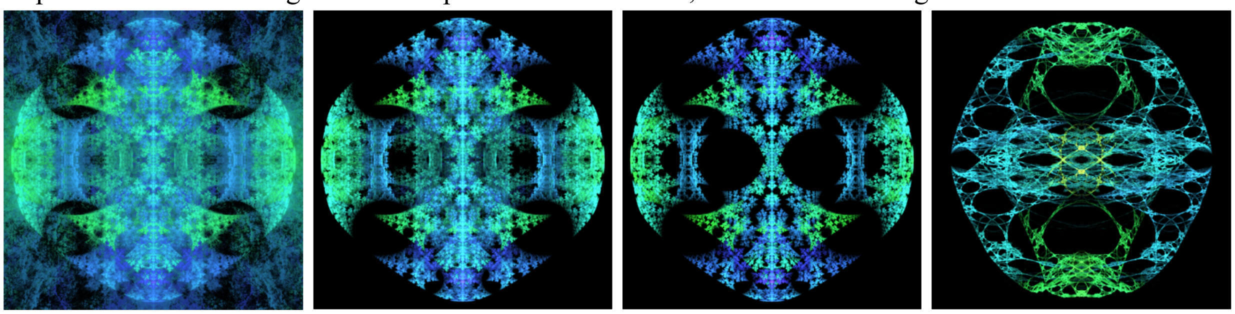
Figure 4: *Inversive fractals with layout identical to Figure 2c except with restricted center circle inversion replaced by restricted inversion in a) ellipse, b) square, c) supershape, d) rhodonea For each accumulated point, color indicates proximity to the boundary of the last inversion applied.*

Gdaweic [6] showed that, like circle inversions, groups of intersecting shape inversions can generate compact fractals if fully restricted limit sets are used. We have also experimented with intersecting shape inversion groups, but applying our new technique of mixed-restriction limit sets instead. Examples are shown in Figure 4 for a configuration of four unrestricted outer circles together with a restricted inner centered star shape that intersects the outer circles at non-kaleidoscopic angles. For each case, if all the inversions were unrestricted the limit set would be noncompact, and if all the inversions were restricted the limit set would be different than in the shown mixed-restriction case (similar to the different limit sets for different restrictions shown in Figures 2a-c for circle inversions).

3D Inversion and 3D Inversive Fractals

Spherical inversion is the natural extension of circle inversion from the 2D plane to 3D space. Groups of inversions in spheres can be used to construct 3D limit sets [4]. If the spheres intersect at kaleidoscopic dihedral angles, iterated inversions can produce a compact fractal limit set [3]. We have explored applying our mixed-restriction technique to intersecting sphere inversions with non-kaleidoscopic dihedral angles. Figures 5a-c shows the limit sets of inversions in an octahedral configuration of six outer spheres with a central sphere intersecting all outer spheres at non-kaleidoscopic dihedral angles. Figure 5a shows that the limit set for unrestricted inversions is noncompact, whereas Figures 5b and 5c show that different mixed restrictions on the inversions yield different compact fractal limit sets.

Ellipsoid inversion is the extension of 2D inversion in an ellipse to 3D inversion in an ellipsoid [5]. As an initial foray into exploring inversion in different 3D shapes we have created mixed-restriction ellipsoid inversions that generate compact fractal limit sets, as illustrated in Figure 5d.



5a) 5b) 5c) 5d)

Figure 5: *Inversive 3D fractals, a) intersecting octahedral group of spheres with unrestricted inversions for all spheres, b) same as 5a but inversion in inner sphere restricted, c) same as 5a but inversion in outer spheres restricted, d) mixed-restriction limit set from a group of ellipsoid inversions*

Source code is available on GitHub as a plugin for JWildfire [10], an open source application for creating IFS-based algorithmic art. Additional information and examples are available at http://genomancer.org.

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