Constructing Deltahedra from Recycled Plastic Bottles

Weiling He
Chengde Wu
Department of Architecture
Texas A&M University

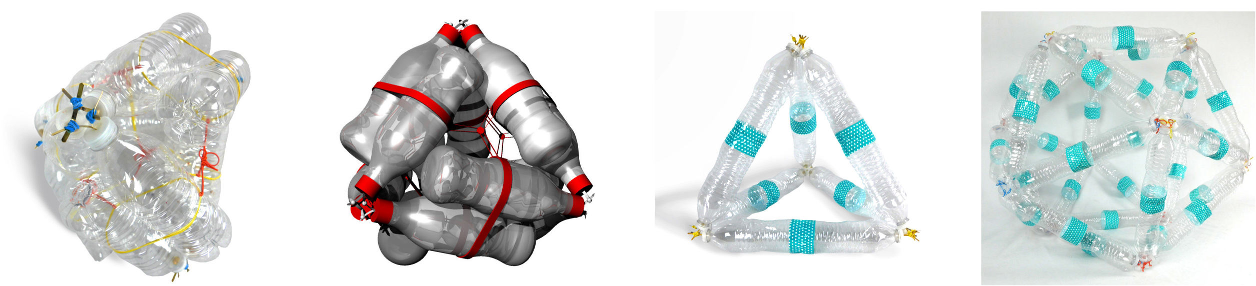
Ergun Akleman
Departments of Visualization &
Computer Science and Engineering
Texas A&M University

Abstract

In this project, our goal is to teach children both recycling and geometry through an integrated process, by constructing deltahedra from recycled plastic bottles. We demonstrate that the two-bottle modules can potentially represent all possible 2-manifold meshes. Moreover, for structural stability, we use triangular faces. All possible deltahedra; i.e., polyhedra whose faces are all equilateral triangles, can be constructed by assembling these two-bottle modular units. And, since we want the process to be as "easy and safe" as possible for children, we avoid cutting the bottles, and join them using common schoolroom materials: tape, rubber bands, etc.

1 Introduction and Motivation

Empty plastic bottles contribute to a significant portion of the by-product of our everyday consumption, but they can be a resource for creativity instead of waste. To utilize these waste products in creative activities, we need to develop processes that can use these plastic bottles for physical construction of 2-manifold meshes.



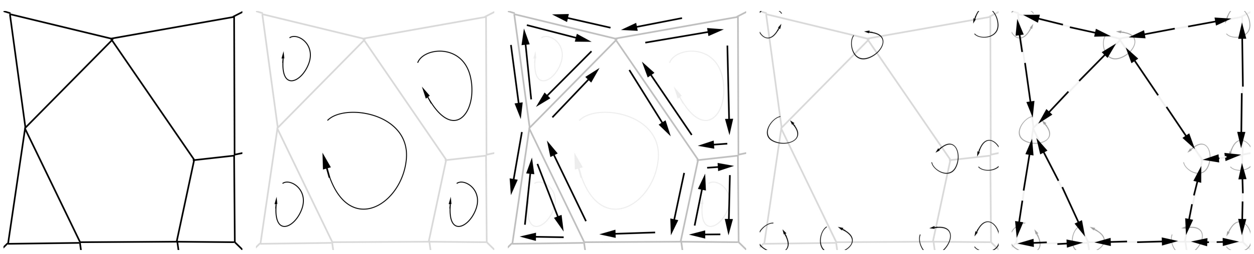
(a) (b) (c) (d)

Figure 1: *Representations of deltahedra that are constructed by using two possible types of two-bottle modules: side-by-side and end-to-end. Figures (a) and (b) show a physical and a virtual tetrahedron built with side-by-side two-bottle modules. Figures (c) and (d) show two regular deltahedra, a tetrahedron and an icosahedron, respectively, that are physically build with end-to-end two-bottle modules. It can be observed in these images that it is harder to construct shapes with side-by-side modules. Additionally, the polyhedral structures are more visible in the shapes that are constructed by end-to-end modules.*

In this paper, we present a process to construct 2-manifold polyhedra that facilitates teaching and learning the concepts related to polyhedral structures and 2-manifold polygonal meshes. Using the same size bottles, these construction processes can obtain any deltahedra that consists of same length edges. Our process is a formalized extension of Mario Marin's construction of icosahedra using recycling soda bottles reported by George Hart [6]. Potentially, this process can become an educational tool to teach young students geometry and design. We intentionally use objects that are easily accessible, such as water bottles, rubber bands, and plastic shopping bags, and are possibly the most common disposable everyday objects. We expect, therefore, that mathematics teachers in the Bridges community can easily employ this process as a barn-raising event [7, 8] for children. The process can also provide a sense of responsibility towards our resources.

2 Methodology and Implementation

Our process is based on two types of directed-edges that are commonly used in computer graphics to represent 2-manifold polygonal meshes [5]. In orientable 2-manifold meshes, the faces and vertices are represented as cyclically ordered sets that are globally consistent as shown in Fig 2. As a consequence of this property, a shared directed-edge appears twice in two opposing directions (See Figure 2) [2]. These two directed components of the same edge can be interpreted as either side-by-side placed, as *half-edges*, or end-to-end placed, as *edge-ends* [2]. Figures 2c and 2d show these two visual representations of directed-edges: *half-edges* (side-by-side) and *edge-ends* (end-to-end). Directed edges are drawn using arrows as shown in Figures 2c and 2d. Any plastic bottle conceptually resembles a directed-edge in the sense that it has a beginning and an ending like a directed edge. In other words, if we view bottles as directed-edges, using two directed-edges, i.e., two plastic bottles, we can construct an edge. The two bottles can be combined in two ways exactly like directed edges: (1) side-by-side, and (2) end-to-end. Based on topological graph theory, two directed-edges are building blocks for representing any 2-manifold polygonal mesh structure; therefore, we can theoretically construct any 2-manifold mesh using these modules by assuming that we can ignore physical constraints such as the material properties or thicknesses of the bottles.



(a) (b) (c) (d) (e)

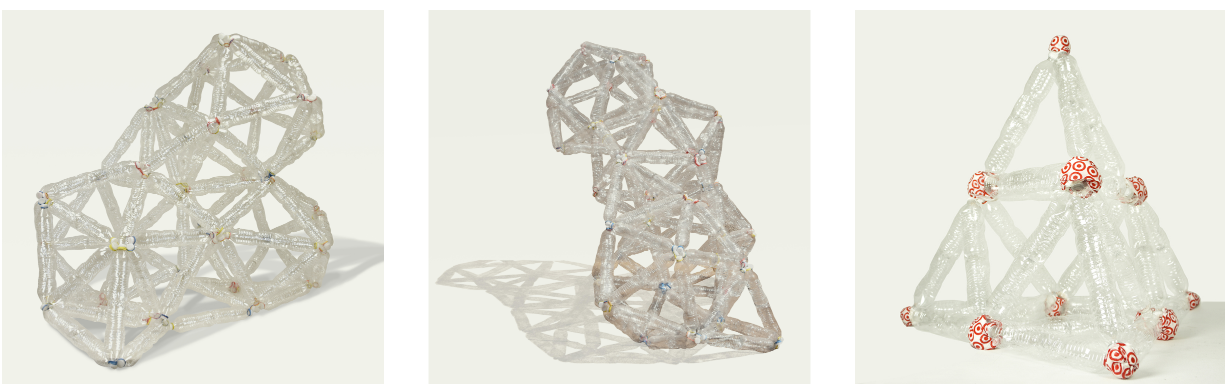
Figure 2: *An example demonstrating the concept of half-edges and edge-ends on a subset of a 2-manifold mesh (see 2a). These concepts come from the fact that an edge consists of two directed edges that form consistent (clock-wise in this case) rotation order on the sides of faces (see 2b) or consistent (counter-clockwise) rotation order around vertices (see 2d). In a manifold, polygonal mesh edges can be separated into two directed edges that can be placed either side-by-side as half-edges (see 2c) or end-to-end as edge-ends (see 2e). These directed edges play an important role to develop consistent data structures for polygonal mesh [1, 2].*

We have experimented with both side-by-side and end-to-end modules and identified that end-to-end are simpler modules with which to work. Moreover, it is easier to recognize the underlying polyhedral structures for the shapes that are constructed by using end-to-end modules as demonstrated in Figure 1. To make one end-to-end bottle module, one starts with turning two bottles to opposite directions and puts them together at their two ends. After the relative positions of the bottles are set, one uses packing tape to tie the two bottles together. In the examples shown in Figures 1c and 1d, we used colorful tape to add an additional aesthetic to the modules. On the other hand, transparent packing tape provides an illusion of two-headed bottles as shown in Figure 4. The length of the edge modules can precisely be controlled by cutting their ends as in Mario Marin's construction of an icosahedral structure [6]. However, to avoid injuries that can result in cutting bottles, we recommend not to cut bottles. For simplicity, we suggest using the same type of bottles throughout one deltahedron design. Therefore, the lengths of the edges in our constructions will always be the same. This constraint restricts possible faces to be regular triangles, since polygons with the same length sides, except triangles, are not uniquely defined. Therefore, we can construct only deltahedral shapes. We can also construct non-manifold structures such as Sierpinski tetrahedron, but, non-manifolds require a higher level representation that does not fit this formalism [3].

Deltahedra appear to be a significant restriction on first glance because there exist only eight convex deltahedral cases [4] with three regular and five Jonhson solids [9]. Fortunately, there exists infinitely many non-convex and non-regular deltahedra such as Stewart toroids[13]. These shapes can be constructed using a method that can preserve edge lengths such as regular face replacements [11, 10] or a regular version of a triangular subdivision [12]. Figure 4 shows a few such irregular examples that we have built as a proof-of-concept. Figures 4a and 4b are obtained by using regular face replacement methods by replacing any triangle of a given deltahedron with either tetrahedron, octahedron, or icosahedron. The structure shown in Figure 4c is a 2-manifold structure that is obtained by a regular triangular subdivision of a tetrahedron, although it can also be considered to be a Sierpinski tetrahedron.

(a) (b) (c) (d)

Figure 3: *These examples show physical connections that represent vertices of polygonal meshes. Each edge-end is represented by caps of the plastic bottles. Vertices are constructed by connecting these caps. We have identified two types of connection mechanisms: Figures (a), (b) and (c) show connectors that play the role of corners by connecting every two consecutive edge-ends of valence-*3*, valence-*5*, and valence-*5* vertices respectively. Figure (d) shows plastic shopping bags that play the role of a 2D thickened valence-*3 *vertex.*



(a) (b) (c)

Figure 4: *Examples of irregular deltahedra. Figures (a) and (b) show two irregular deltahedra that are constructed by architecture students in a barn-raising event. Figure (c) shows a subdivided tetrahedron.*

To construct 2-manifold mesh structures, we need to connect *edge-ends* to form vertices as shown in Figure 3. The method of connecting *edge-ends* plays a crucial role in guaranteeing that final structures are true representations of 2-manifold meshes, since the connections should describe vertex neighborhoods that are homeomorphic to the 2D Euclidean space (i.e. a disk). This means, in practice, that the vertex should be thickened into a polygon [5]. We identified two methods for thickening. The first method is based on the fact that connecting every two neighboring *edge-ends* of a valence-*n* vertex results in a well-defined polygon (See Figures 3a, 3b, and 3c). The methods in Figure 3a, 3b, and 3c requires one, two and no slit on the cap respectively. We observe that even though the methods shown in Figures 3a and 3b result in more stable structures, since they require slitting the center of caps, they can be dangerous for children. We, therefore, suggest using rubber band connectors shown in 3c to construct vertices with this method (See Figures 4a and 4b). The second method is to directly construct vertex neighborhoods that are homeomorphic to 2-space by using plastic shopping bags. In this method, one simply puts plastic shopping bag between the cap and the bottle and screws the cap on. Since shopping bags are very thin, screwing caps automatically creates a connection (See Figure 3d).

3 Discussion and Future Work

We have built a variety of deltahedra using these two types of connectors. We observed that shopping bag connectors are easy to use (See Figure 4c); however, it is hard to make large structures with shopping bags since they do not provide enough rigidity within the structure. For larger structures such as the ones shown in Figures 4a and 4b, the first method with rubber bands is more practical. This work was partially supported by the National Science Foundation under Grants NSF-EFRI-1240483, NSF-CMMI-1548243 and NSF-ECCS-1547075.

References

[1] Ergun Akleman and Jianer Chen. Guaranteeing the 2-manifold property for meshes with doubly linked face list. *International Journal of Shape Modeling*, 5(02):159–177, 1999.

[2] Ergun Akleman, Jianer Chen, and Jonathan L. Gross. Paper-strip sculptures. In *Shape Modeling International Conference (SMI), 2010*, pages 236–240. IEEE, 2010.

[3] Ergun Akleman, Jianer Chen, and Jonathan L. Gross. Block meshes: Topologically robust shape modeling with graphs embedded on 3-manifolds. *Computers & Graphics*, 46:306–326, 2015.

[4] Henry M. Cundy and Arthur P. Rollett. *Deltahedra: in Mathematical Models*. Tarquin, 1981.

[5] Jonathan L. Gross and Thomas W. Tucker. *Topological graph theory*. Courier Corporation, 1987.

[6] George Hart. Math monday: Recycling soda bottles into icosahedra, May 2015.

[7] George W. Hart. Icosahedral constructions. *Proceedings of Bridges: Mathematical Connections in Art, Music and Science, Southwestern College, Kansas*, 1998.

[8] George W Hart. Sculpture based on propellorized polyhedra. *Proceedings of MOSAIC*, pages 61–70, 2000.

[9] Norman W. Johnson. Convex polyhedra with regular faces. *Canandian Journal of Mathematics*, 18(1):169–200, 1966.

[10] Eric Landreneau, Ergun Akleman, and John Keyser. Interactive face-replacements for modeling detailed shapes. In *International Conference on Geometric Modeling and Processing*, pages 602–608. Springer, 2006.

[11] Eric Landreneau, Ergun Akleman, and Vinod Srinivasan. Local mesh operators: Extrusions revisited. In *Shape Modeling and Applications, 2005 International Conference*, pages 349–354. IEEE, 2005.

[12] Charles Loop. Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, United States, 1987.

[13] Bonnie M. Stewart. *Adventures among the toroids*. BM Stewart, 1980.