# Cut Colored Paper Sculptures of 3D Contour Plots of the Real and Imaginary Parts of Complex Functions

Caroline Bowen

Eigenstuff

PO Box 837, Louisville, TN, 37777, USA

`cb@eigenstuff.com`

## Abstract

I have created a series of sculptures depicting 3D contour plots of the real and imaginary parts of complex functions made from die-cut colored cardstock, acrylic rods, and acrylic spacers. The graphs were plotted as 2D contour plots in Mathematica, formatted for cutting in Adobe Illustrator, cut from colored scrapbooking paper using a Cameo Silhouette paper-cutting machine, and assembled. They evolved out of a previous project, creating cutout concertina cards illustrating MacLaurin series and special functions.

## Introduction

I became interested in working with cut paper after wanting to explore the use of pop up books as a cheap, portable format for 3D illustrations of concepts in undergraduate level math and physics. In June 2016, after six months of practicing with my X-Acto knife and experimenting with pop-up spread ideas, I started making cutout concertina cards illustrating different MacLaurin series expansions, Fourier expansions, and special functions. I then decided to see what would happen if I took this same idea and
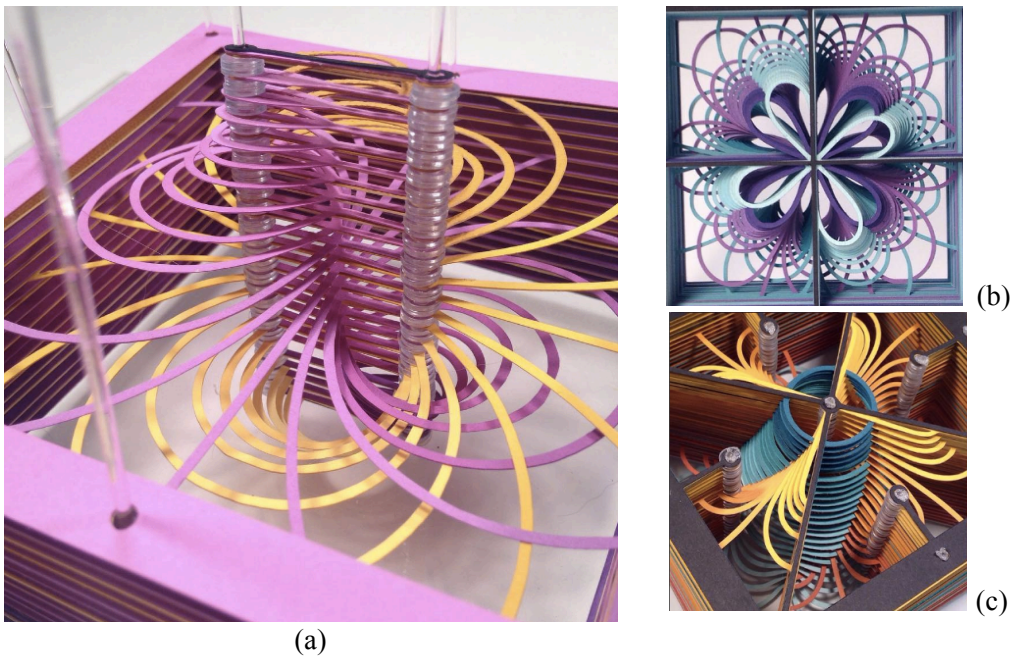


**Figure 1:** *(a)* Sech(z)*, real part purple, imaginary part yellow. (b)* Csch($z^4$)+ArcSech($z^2$)*, real part teal, imaginary part purple. (c)* ArcSech($z^4$)*, real part teal, imaginary part orange*

applied it to 3D contour plots by taking 2D contour plots and giving each level more space between its neighbors. Soon I realized that this method limited me to functions whose graph lines connected with the bounding box−there could be no elements free floating in the middle. I then very quickly ran out of functions to make that were interesting enough to be worth the effort. I then began to experiment with making these same cutout contour graphs mounted on rods with spacers in between. Initially, I used paper spacers cut out using my die cutting machine and used wooden dowel rods mounted onto pieces of basswood, but eventually I moved to using 1/16" acrylic rods and small acrylic spacers meant for use in computer hardware. Up to this point, all of the paper sculptures I had made were of real functions, but I decided to see how it would look if I did one of the real and imaginary parts of a complex function together. For each point in the complex plane, a complex function *f(x+iy)=w=u+iv* is a complex number with a real part *Re[w]=u* and an imaginary part *Im[w]=v*, each of which effectively generate a 4D surface that are typically graphed as separate, static 3D surfaces. Here, I have interwoven both of these 4D surfaces together by illustrating them as, in a way, a 3D timelapse of MRI-like projections of the 4D surface onto the parallel, equidistant 2D planes at constant height above the complex plane. But in order for this to work, I had to switch from cutting out whole areas for each contour level and instead cut out the graph lines so the two parts wouldn't obscure each other from view.

I was very pleased with how my first attempt turned out (made with only black and white paper), and then realized I could very easily remove the entire paper piece from the wooden base by simply melting and mashing down the ends of the acrylic rods. This then enabled me to view the piece from both sides, each one revealing a different pattern. However, up until now, all of my colored cardstock gradients were continuous: each contour level was a different color from its neighbor. But this was not practical for these two-part sculptures because of the limits on what paper colors I had available to me, so I began making them with two gradients made up of five different shades each, each shade covering five contour levels for a total of twenty-five.
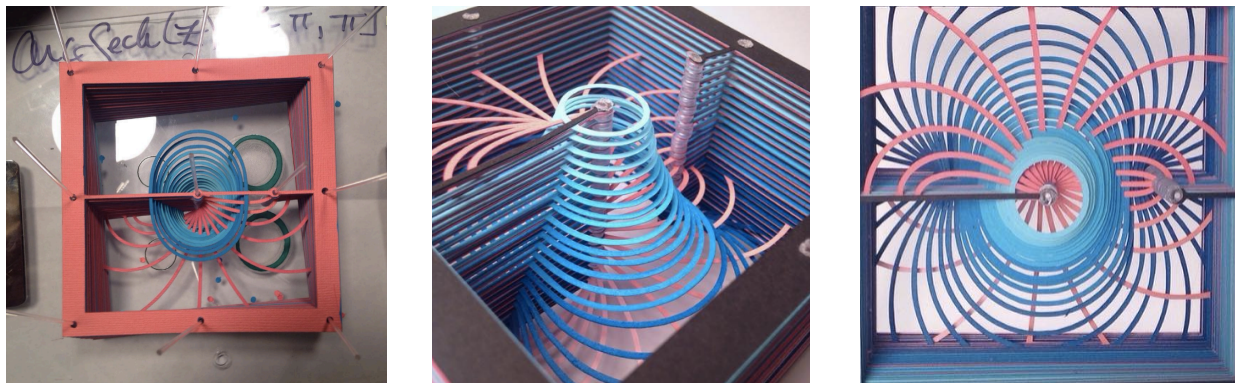


**Figure 2:** ArcSech(z)*, real part blue, imaginary part pink. The first image is of the sculpture half assembled; the second two are of the completed piece.*

## Construction

I found visually interesting functions to make into sculptures by effectively banging my hands on my keyboard: I would plot a completely ridiculous and impractical complex trig function in Mathematica, usually the sum of two terms, and then tweak it by changing the exponents of the arguments and adding or removing an 'h' to make the trig functions hyperbolic or regular. When I found a function I liked enough to spend several hours making it out of paper, the real and imaginary parts were plotted separately as 2D contour plots in Mathematica, their graph lines were converted to paths, and then the plots were exported as .`svg` files. These were then opened up in Adobe Illustrator, where the paths were sorted into layers by contour level, the graph line thickness increased to 4.5 to 4.25 pt, and then the bounding box

frame template was copied and pasted underneath the graphs. If there were branch cuts in the function, I would draw those in manually using the line tool, and at the end of each one, draw in a circle with a hole in it that would be used to attach the branch cut to a supporting rod. Then, I went through the list of layers, turning them all off except for one, and exported that as a .png. I would also turn off all the contour layers but leave on the layer containing the branch cuts and exported this, as this served as the black border on the top and bottom of the sculpture as well as visually distinguish the branch cuts from the actual graph lines. The .png files were then opened up in Silhouette Studio, the proprietary software for my paper cutting machine (Cameo Silhouette), and cut out of paper.
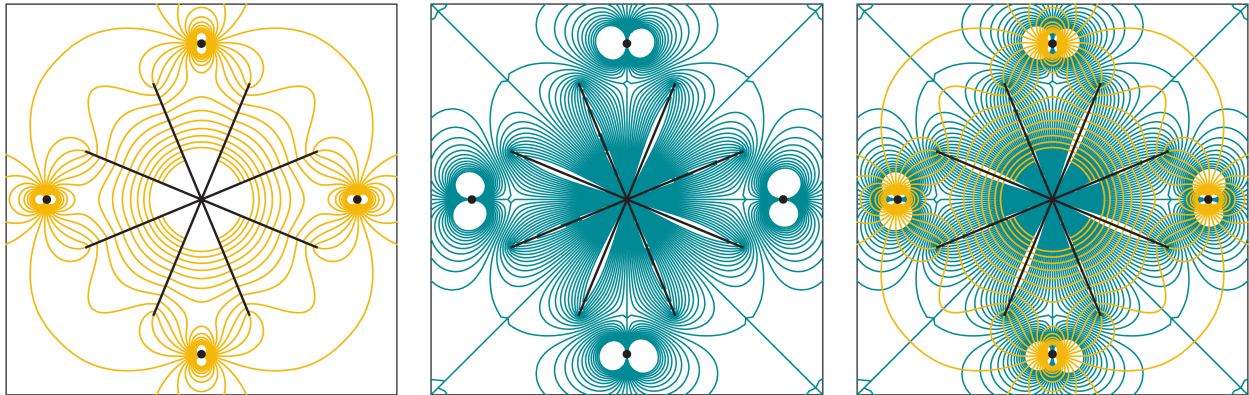


**Figure 3:** ArcCsch($z^4$)+Sec($z^2$): *plots of the real part, imaginary part, and both parts together.*

Assembling the colored paper gradients was the most difficult part because I was at the mercy of scrapbook supply companies as far as my color options were concerned (and orange, in particular, was almost impossible to make a gradient for). I spent several hours in the paper aisles of Michael's and Jo Ann's stores, looking at different combinations of paper colors, and trying to find five-color gradients whose colors were all the same hue and saturation as well as distinguishable enough from one another that the steps between each color in the gradient were equal. I made some attempts to dye paper myself to achieve the color gradients I wanted, but I found that the white cardstock I was using did not take the dye well enough to achieve the rich, saturated colors I wanted. With a few exceptions, I had a general rule that I would use cool colors for the real parts and warm colors for the imaginary parts, my reasoning being that the real numbers are stationary and static (thinking of the real number line) while in comparison the complex numbers are much more dynamic and energetic, having to deal with rotations and stretching. I also decided to have one color gradient going from light to dark and the other one going from dark to
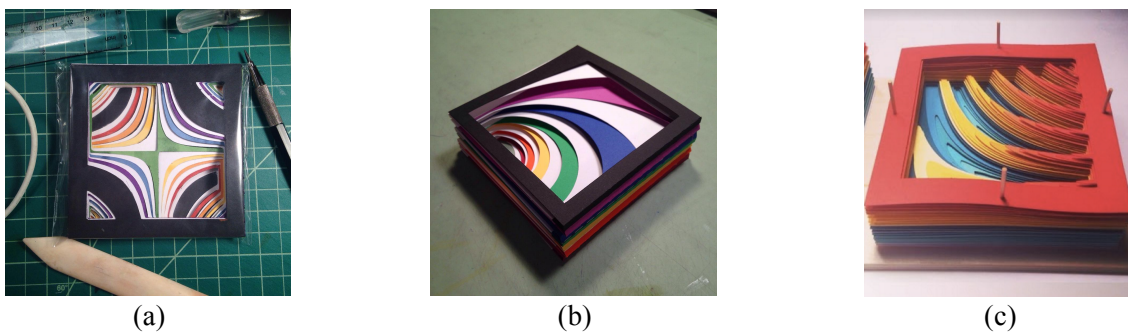


| (a) | (b) | (c) |

**Figure 4:** *Some early predecessors. (a) Hand-cut concertina card of* Sin(xy). *(b) Machine-cut concertina card of* ArcTan(z). *(c)* xSin(xy), *with the contour levels mounted on wooden rods.*

light, as I felt this added an additional bit of visual flair compared to simply having two gradients going in the same direction. To begin assembling the sculptures, I first cut out a drill guide using my paper cutting machine that I then taped to a piece of scrap Plexiglass, drilled holes for the acrylic rods using a hand drill, and drew a star by the top left corner of the drill guide to help keep myself properly oriented during assembly. I would then light one end of the rod on fire, let it burn for a few seconds, blow it out, and then mash it down with my lighter to keep it from falling out before inserting it into the hole. Once all the rods were inserted, I would then take another piece of Plexiglass and clamp it underneath the first one to hold the rods in place. Next, I began assembling the sculptures by sliding the cut out contour levels onto the rods and putting two spacers on each rod between each level. Once assembly was complete, I melted and mashed in the tips of the rods, flipped the sculpture over, and cut off the burn part of the bottom of the rods in order to slide the scrap Plexiglass base off. Then, I used an eraser guard to hold down the paper around each rod and then trimmed it, melted and mashed the end down.
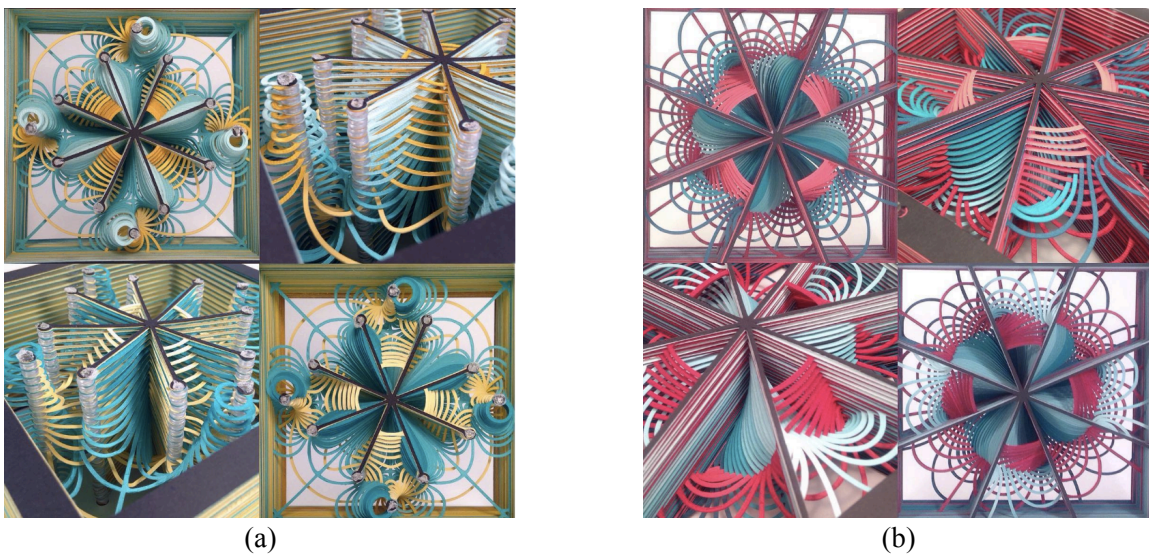


(a)            (b)

**Figure 5:** *(a)* ArcCsch($z^4$)+Sec($z^2$)*, real part yellow, imaginary part teal.*
*(b)* ArcTanh($z^4$)+iArcCsch($z^4$), *real part teal, imaginary part red.*

## Going Forward

After making nineteen of these over the course of four months, I have moved on to other endeavors. However, it would not be difficult to scale up these sculptures from 4.5" square to at most 12" square without having to significantly alter the construction process. My primary reason for not having done so has to do with costs: scrapbooking paper is very expensive. I have also found limitations on what my computer is capable of plotting that would prevent me from scaling up the number of contour levels for more complicated graphs. I am interested in one day revisiting this project and figuring out how I could apply it to illustrating concepts in physics.

## References

[1]  T. Needham. *Visual Complex Analysis*. Oxford University Press. 2000.
[2]  E. Wegert. *Visual Complex Functions*. Birkhäuser. 2012.
[3]  J. Brown and R. Churchill. *Complex Variables and* Applications. McGraw-Hill. 2013.