A Temari Permutation Sampler

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Abstract

We discuss the mathematics and the craft involved in creating a temari ball embroidered with 24 trefoil knot designs. Each design includes four colors of thread permuted in a different order. The designs are arranged to form a truncated octahedron, where designs formed by switching adjacent colors are connected by an edge.

Introduction

Below are three views of an embroidered thread ball with diameter 15 cm. The 24 four-colored trefoil knots are arranged so that each pair of knots connected by a silver guideline thread has one pair of adjacent colors switched.

Figure 1: *Three views of the Temari Permutation Sampler Ball. Moving clockwise and listing colors—light green (1), light purple (2), dark green (3), dark purple (4)—from inside to outside of the designs, the left view shows the square 1234-1243-2143-2134. The middle view shows the hexagon 1432-1342-3142-3412-4312-4132. The right view shows the vertex 4231 and its neighbors.*

The text below describes both the math and the craft of making this ball.

The Permutation Dance Activity

Many years ago when teaching Discrete Math I made up a "Permutation Dance," activity, where four students stand in a line. One pair of students standing next to each other switches places. The goal is to continue switching and exhibit all 24 permutations of the students, without repeating any, and ending in the initial position. This activity is similar change ringing, where all permutations of a set of church bells are rung in sequence, under slightly different constraints [1].

The physical part of the activity is more helpful for describing the problem than for solving it. In looking for a paper and pencil solution, students sometimes make a graph showing all possible switches; the Permutation Dance is a Hamiltonian Cycle in the graph, which visits each vertex once. The first pass at such a graph might be quite a bit messier than the graph in Figure 2, where each vertex represents a permutation of the students, and the underlying structure is a truncated octahedron. Each vertex of the truncated octahedron joins two hexagons—one showing a cycle of permutations with the first digit fixed and one showing a cycle with the last digit fixed—and one quadrilateral, which shows a cycle that alternates switching the first two and the last two digits. This graph is the Cayley graph of the permutation group S4, generated by the switches (12), (23), and (34).

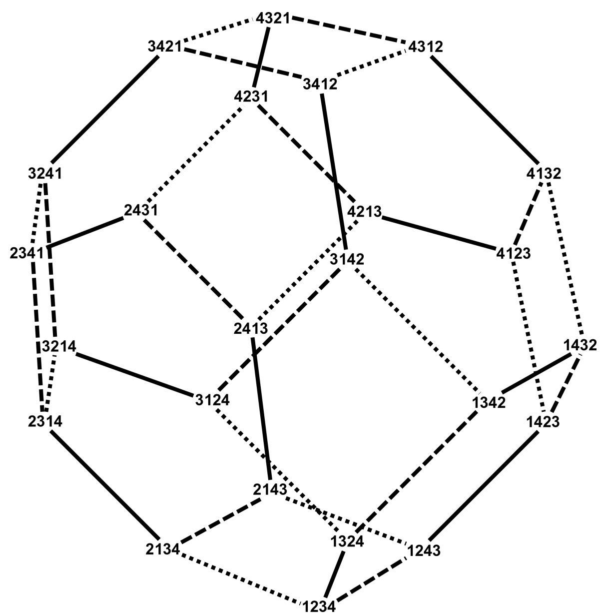

Figure 2: *The truncated octahedron structure of the graph in Figure 2. The dotted, solid, and dashed edges represent respectively, when the first, middle, and last two people switch places.*

Making the Temari Permutation Sampler Ball

I first learned of the Japanese/Chinese art of temari from the picture of Carolyn Yackel's work on the cover of *Crafting by Concepts* [4]; I immediately wanted to learn to make them myself. Later I recognized temari as a fiber craft that could be used to show the graph in Figure 2, with four colors representing the permutations. The temari literature uses less mathematical language, but it includes all Platonic Solids and some Archimedean Solids among its standard markings.

There were several tricky parts to creating the ball: making a ball big enough so that the 24 copies of the design would clearly show all four colors, figuring out how to mark a truncated octahedron on the ball, choosing a design that would work with the geometry of the ball and that was within my skill level, and choosing the colors.

Constructing the ball. I began with a plastic bag filled with about seven cups of rice hulls (the remains after brown rice is refined to white rice) and followed a procedure adapted from the TemariKai website [3] that I've used before for smaller balls. I wrapped some yarn around the bag to hold it together, covered everything with quilt batting, then with more yarn, then with multiple strands of serger thread, then with a single strand of thread.

This ball was about the maximum size I could hold in one hand, and I often dropped it when I was winding the thread. The final product was a little too hard and too heavy, with the final wrap not as thick as on smaller balls I've made, but overall the ball worked well as a base for the design. For the next version, I will make the ball a little smaller and use a lighter filling.

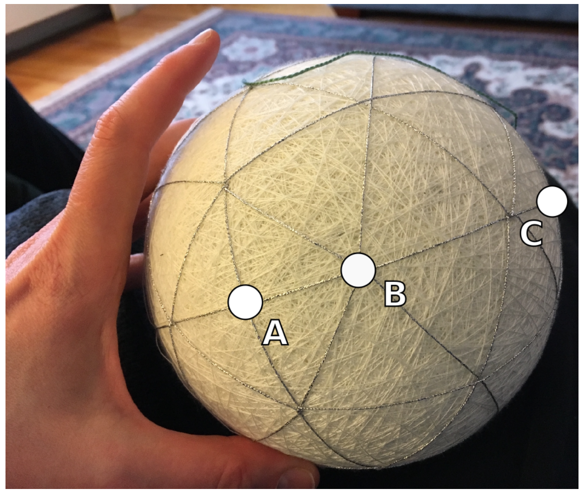

Figure 3: *The ball marked with silver guideline threads*

Marking the ball. To mark guidelines for the geometry of the ball I started by marking an octahedron, with three perpendicular great circles (an S4 marking in temari vocabulary). Two of these great circles intersect at point A in Figure 3, and two also intersect at point C. The great circles intersect at six vertices, and we call these vertices $(0,0,\pm 1)$, $(0,\pm 1,0)$, $(\pm 1,0,0)$, with $A=(0,-1,0)$ and $C=(1,0,0)$.

To truncate the octahedron, we separate the Euclidean edge $\overline{AC}$ into thirds; the extra vertex closest to A is (1/2, -2/3, 0). This point projects onto the sphere at $B = (1/\sqrt{5}, -2/\sqrt{5}, 0)$. To find the central angle θ, of $\overset{\frown}{AB}$, we take the dot product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and find that $\theta = \cos^{-1}(2/\sqrt{5}) \approx 26.57°$. Thus $m\overset{\frown}{AB} / m\overset{\frown}{AC} \approx 26.57/90 \approx 0.295$, which worked out to measuring 3.5 cm from A to mark B with a pin.

I marked all 24 points analogous to B, and sewed in the squares. Then I sewed the missing long diagonals of the hexagons to create additional guidelines for the design elements. Note that temari artist Barbara Seuss's directions for a "14 Faces Method #1" marking [2, page 47] are virtually identical, but I wanted to understand the derivation.

Design elements. I chose the temari tri-wing design [2, pages 117-119] because it's formed on three guidelines, and three polygons meet at each vertex of the truncated octahedron. It's easy to make the design a bit asymmetric to match the different lengths of the guidelines that are diagonals of hexagons and squares. I made the two balls in Figure 4 to practice the design elements, and they also look nice on their own.

Color. I chose a neutral color for the ball, and then choosing the thread colors was surprisingly difficult. I wanted four distinct colors that would look good together and that were available in Perle Cotton Thread. I ended up picking light and dark variations of two colors because I liked the way they matched, but I was surprised at how different neighboring designs could look – especially when a light and a dark color switched. The ball is very useful as a sampler to see how the order of the colors affects the final look, but I had been hoping for a more subtle effect that might say something more about the mathematical structure. Perhaps a different choice of colors would achieve such an effect; perhaps there is just too much visual information to take in. I hope others will experiment with using different mediums to see what visual effects are possible in a permutation sampler.

Figure 4: *Two smaller (7 to 8 cm diameter) preliminary balls to learn to sew the ball (left) and see how the pattern worked on the whole ball (right).*

More Mathematical Properties

We now return to the original problem of finding the "Permutation Dance," which students often try to solve by keeping the first digit fixed for six permutations in a row. Geometrically, this strategy involves traversing four of the hexagons; the centers of these hexagons form a tetrahedron. On the model it's fairly straightforward to see that we can easily traverse three of the hexagons, but then we end in a place where we can't get to the remaining hexagon. A more fruitful strategy is to look at the squares, which form a cube whose faces are separated and rotated 45°. We traverse complete squares, and the decision whether to traverse clockwise or counter-clockwise determines the subsequent square. The reader is invited to try using this strategy to try to find a Hamiltonian Cycle in Figure 2.

Handling the ball also reveals visual versions of interesting mathematical properties. For example, in two neighboring hexagons, one has the innermost thread fixed and one the outermost. Design elements that are a 180° rotation of the ball away from each other have colors in the opposite order. Design elements reached by travelling across two square diagonals (and the hexagon edge between them) have the first and last color switched. The white space forms interesting star patterns.

References

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