Hyparhedra Revisited

Eve Torrence
Department of Mathematics
Randolph-Macon College
Ashland, VA 23005, USA
etorrenc@rmc.edu

Abstract

In their 1999 Bridges paper, "Polyhedral Sculpture with Hyperbolic Paraboloids," Erik Demaine, Martin Demaine, and Anna Lubiw discussed gluing together origami "hypars" to form structures they called "hyparhedra." By experimenting with this method, I discovered that using a rhombic triacontahedron instead of a dodecahedron as the underlying polyhedron for the "hypar dodecahedron" results in a more symmetric model. The hypars are folded from rhombic paper instead of squares, with the size of the rhombic paper determined by experimentation. Three of the resulting structures, "Day," "Night," and "Pahoehoe," were included in the Art Exhibits at the Bridges Conferences in 2015 and 2016 and "Day" won the Best of Show People's Choice Award in 2015.

Introduction

A hyperbolic paraboloid is a surface that resembles a saddle, see Figure 1. A nice model that approximates this shape can be made by folding a square piece of paper, see Figure 2. Directions and diagrams for folding origami hypars can be found in [1]. Throughout this paper I will use the term hypar to refer to origami version of this shape.

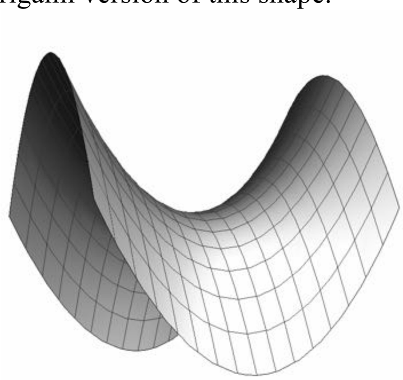

Figure 1: *A hyperbolic paraboloid*

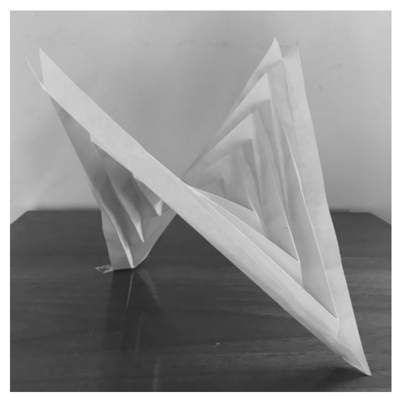

Figure 2: *An origami hypar*

Hyparhedra are formed by glueing hypars together to form structures with underlying polyhedral symmetry. Demaine et al's method focuses on the polygonal faces of the polyhedron to be modeled. For each *n*-gon face they glue *n* hypars together to form a polygonal "hat." These hats are then glued together to form a "hyparhedra" with each "hat" corresponding to a face of the polyhedron being modeled.

This method works beautifully to form a "cube" but produces a lopsided model when applied to the dodecahedron. For example, compare the hypar cube and the two hypar dodecahedra in Figure 3. The square faces of the "cube" are formed by four hypars in alternating colors. The pentagonal faces of the "dodecahedron" are formed from five hypars, one of each color. The hypar cube is a stable symmetric structure but in the two dodecahedra some points where the hypars meet are convex and others are concave and the vertices of the pentagonal faces do not lie in a plane.

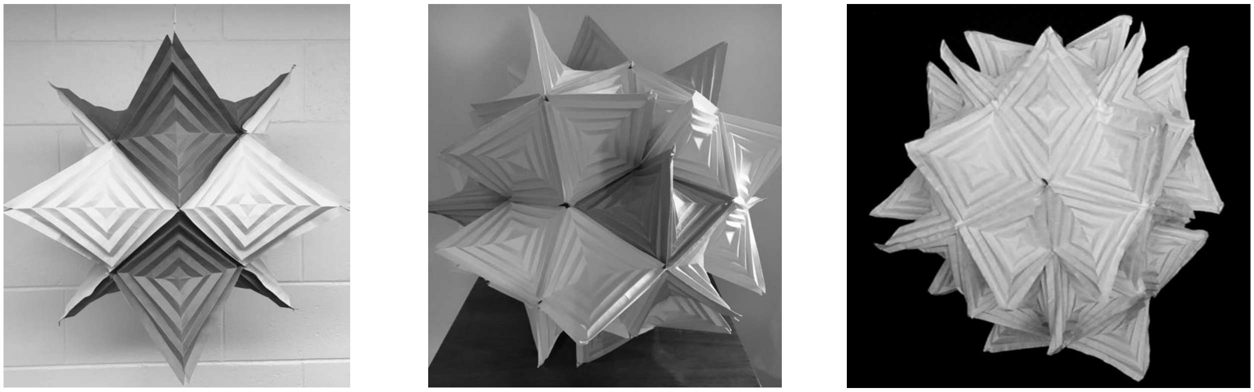

Figure 3: *A symmetric hypar cube (left) and lopsided hypar dodecahedra by the author (center) and by Demaine et al from [1] (right).*

We Need a Different Foundation

Why can we make a nice hypar cube but not a nice hypar dodecahedron? My breakthrough in solving this problem was to realize that these structures had more than one underlying polyhedron. To understand why the hypar cube is symmetric we must first realize the underlying structure can also be thought of as a rhombic dodecahedron, see Figure 4. When two hypars are glued along two edges, the four outside edges will lie on a rhombus. One such rhombus is outlined in the image in the center of Figure 4. The ratio of the length of the long diagonal to the short diagonal of this rhombus is approximately $\sqrt{2}$. The faces of a rhombic dodecahedron are rhombi with this ratio and this is the ratio we get when the hypars are folded from square sheets of paper.

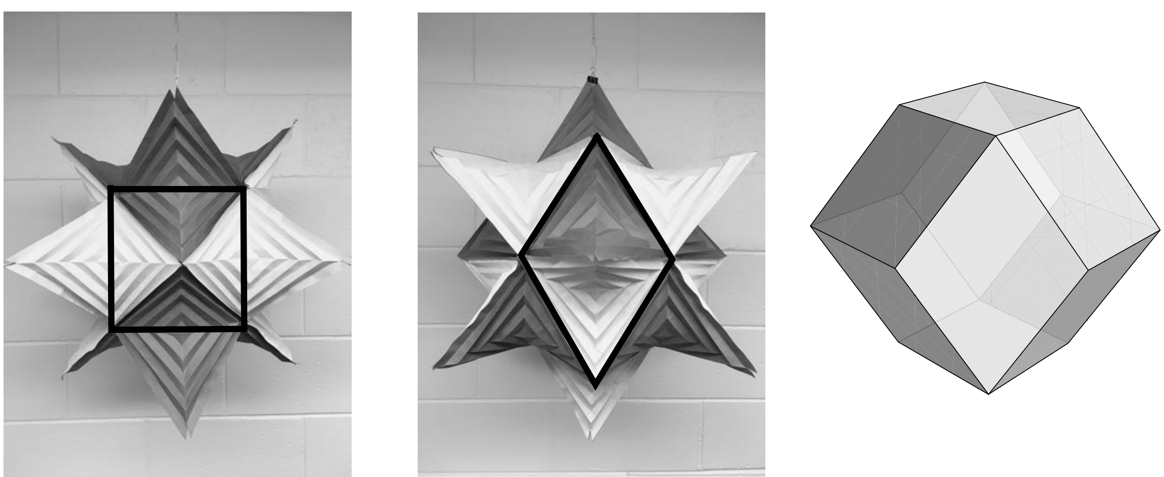

Figure 4: *The same structure is shown from two different angles with a square face of the cube emphasized (left) and a rhombic face of the rhombic dodecahedron emphasized (center). Compare the center image with the rhombic dodecahedron (right).*

Thinking about the geometry of the underlying rhombic structure allowed me to create a hypar dodecahedron that is symmetric. Instead of thinking of the hypar dodecahedron as being made of pentagonal faces created from five hypars I focused on the structure of a rhombic triacontahedron for the underlying geometry. In Figure 5 the similar symmetry of the dodecahedron and the rhombic triacontahedron can be seen. The rhombic triacontahedron is the dual of the icosidodecahedron and the zonohedron of the icosahedron. All of these solids have icosahedral symmetry.

A rhombic triacontahedron is made from 30 rhombi with a ratio of long diagonal to short diagonal equal to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. The golden ratio is greater than $\sqrt{2}$ so these rhombi are slightly narrower than those of the rhombic dodecahedron. This is why Demaine et al's hypar dodecahedron bulges unsymmetrically; the rhombi are too wide.

In order to model a rhombic triacontahedron I needed rhombi with the correct ratio of diagonals. Square paper does not work. Through experimentation I found that starting with rhombic paper with a diagonal ratio of 1.2 would result in pairs of hypars forming a rhombus with the correct diagonal ratio of φ. The resulting hyparhedron shown in Figure 6 can be thought of as being based on a dodecahedron or a rhombic triacontahedron.

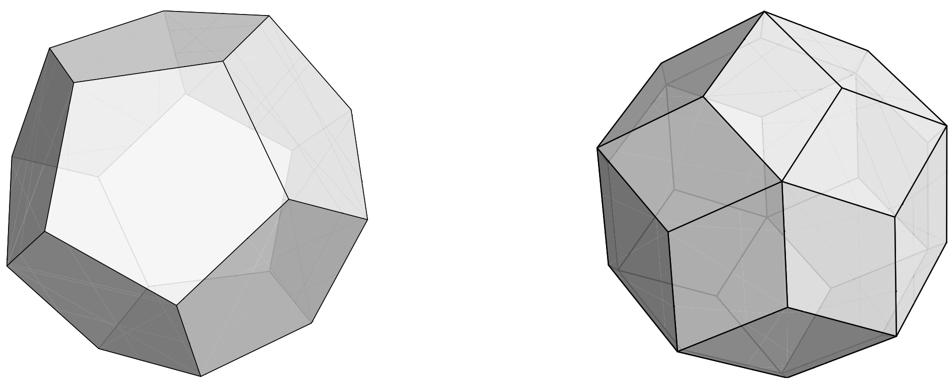

Figure 5: *Compare the symmetry of a dodecahedron (left) to a rhombic triacontahedron (right).*

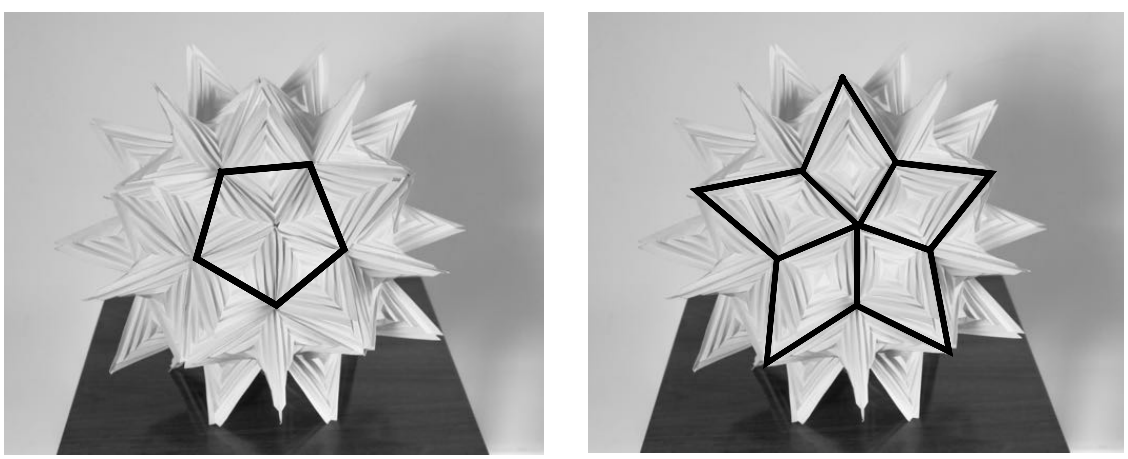

Figure 6: *"Day" can be thought of as being based on a dodecahedron (left) or a rhombic triacontahedron (right).*

I named the resulting sculpture "Day." While building this model I noticed interesting patterns on the inside of this structure and was curious what it would look like if I reversed the orientation of all the units. The result was the sculpture "Night" that is pictured in Figure 7. I was concerned there might not be room for all the "points" on the inside of the structure but they fit, with very little room to spare.

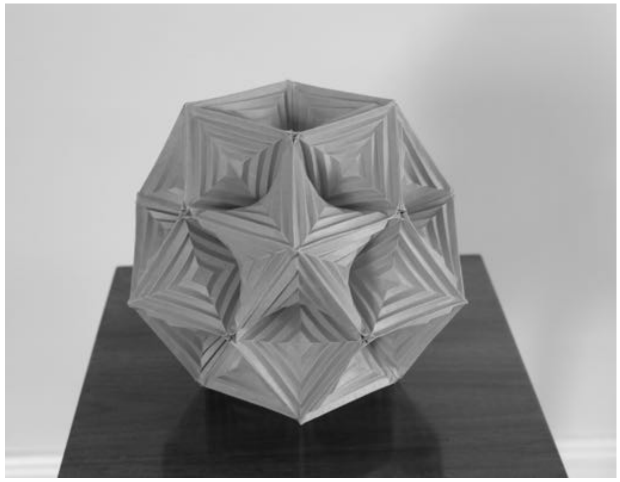

Figure 7: *To form "Night" I faced the points of pairs of hypars toward the center. This creates an interesting effect and makes the rhombic triacontahedron structure more obvious.*

Once I made a symmetric model from sixty hypars folded from rhombi I was excited to see what else I could make. I had already found the correct size paper from which to make hypars that in pairs formed rhombi with a diagonal ratio of φ. It made sense to next try to make another polyhedron formed from these rhombi. A hexecontahedron is a non-convex polyhedron formed from 60 golden ratio rhombi. The resulting hyparhedron, formed from 120 hypars is shown in Figure 8.

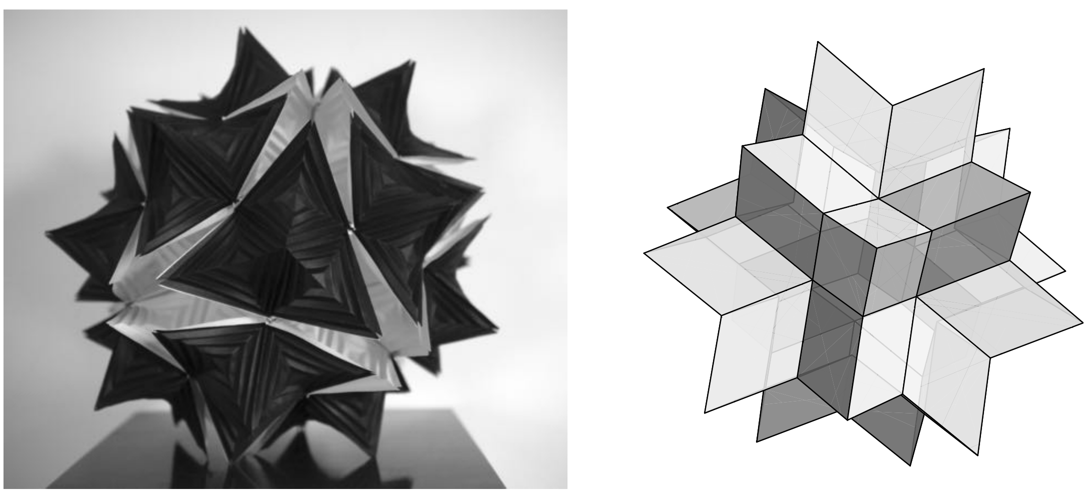

Figure 8: *The structure of "Pahoehoe" (left) is based on a rhombic hexecontahedron (right). Each rhombus is formed from one orange and one black hypar. Pahoehoe is a type of Hawaiian lava that forms parallel ridges when it cools.*

Reference

[1] Erik Demaine, Martin Demaine, and Anna Lubiw. Polyhedral Sculpture with Hyperbolic Paraboloids. In Reza Sarhangi, editor, *Proceedings of Bridges 1999: Mathematical Connections in Art, Music, and Science*, pages 91–100, White Plains, Maryland, Gilliland Printing. Available online at `http://archive.bridgesmathart.org/1999/bridges1999-91.html`