The Golden Ratio: How Close Is Close Enough?

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Abstract

When looking for evidence of the golden ratio in works of art, architecture or music, what tolerance should we apply? I argue that the answer depends on what type of object we are testing, how we are measuring, and on a threshold that is not yet known but is introduced here as the Ratio Just Noticeable Difference (RJND). Finally, we describe a variety of experiments that can be used to establish RJNDs for testing evidence of the golden ratio.

Introduction

I teach a course on Math and the Creative Arts in which I ask students the following:

Using the portrait provided, investigate whether or not the eyes are placed on or close to the horizontal line that cuts the canvas into the golden ratio (with the greater part below the eyes).

This exercise helps students think about the many thorny issues that arise when looking for evidence, if any, of the golden ratio. We must address questions such as: How much evidence is required to conclude that the golden ratio is present in a work of art? Is such evidence coincidental? If not, how do we know if the artist used it with a full awareness of its numerical value and meaning, or if the artist happened upon it by an aesthetic instinct, perhaps because the ratio truly is one of the most pleasing to an artistic eye?

These are just a few of the questions that make the extent of the golden ratio's appearance in art a contentious issue, although its presence in nature is generally accepted (see [6] pg.122). The purpose of this article is not to weigh in on this debate, but merely to expose and clarify some issues, mathematical and otherwise, that are relevant in order to discuss it in an unbiased and critical way.

How Close is Close Enough?

Given a line segment subdivided into two parts, we say that it is cut in the *golden ratio* ϕ if the ratio of the greater to the lesser part equals the ratio of the whole to the greater. A rectangle is a *golden* rectangle if the ratio of its length to width equals ϕ. The first definition leads to a quadratic equation, which we can solve to obtain a value for $\phi = \left(1+\sqrt{5}\right)/2 \approx 1.618$. No amount of care will permit an artist to reproduce any number exactly, rational or otherwise, so unless an artist has stated an intention to use the golden ratio, most arguments that ϕ appears begin with both a measurement (with its own error) and a tolerance, whether or not this tolerance is stated explicitly. This leads us to the question: what range of values around ϕ should we accept, so that any ratio r that falls within this interval provides preliminary evidence that the golden ratio is in play? We will call such an interval a *golden interval*.

Our goal is to introduce criteria for selecting a golden interval *G*, with the intent that if a ratio *r* measured in a work of art is not contained in *G*, then *r* should not be used to argue a case for ϕ. Note that we are not asserting the converse: if such a ratio *r* is contained in *G*, this alone is not conclusive evidence that the golden ratio has been used intentionally, nor that it is aesthetically pleasing to the artist or to the general public. There are many questions and objections that need to be addressed before a claim of this sort can be substantiated. It is not within the scope of this paper to address these.

Before we discuss how to obtain a golden interval, it is essential to realize that when we evaluate how close a line cut is to the golden ratio, the ratio of greater to lesser must be treated differently from the ratio of whole to greater. For example, if a line is cut and the lesser and greater parts measure 1 and 1.6 respectively then the error in the ratio of greater to lesser is $|1.6-\phi| \approx 0.018$. The corresponding measure of the whole is 2.6. Thus, the error in the ratio of whole to greater is $|1.625-\phi| \approx 0.007$. This is because the sequence *lesser*, *greater*, *whole* satisfies the Fibonacci recurrence relation $a_n + a_{n+1} = a_{n+2}$. It is well-known that if we begin with any sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the Fibonacci recurrence relation, then the sequence of ratios of consecutive terms satisfies $\lim_{n\to\infty} a_{n+1}/a_n = \phi$. It can be shown that the errors $|a_{n+1}/a_n - \phi|$ in the approximation of ϕ decrease monotonically as $n \to \infty$. Thus, for any subdivision of a line,

$$\left|\frac{whole}{greater} - \phi\right| < \left|\frac{greater}{lesser} - \phi\right|$$

Consequently, a golden interval is specific to only one of these two ratios, and care must be taken when testing the "goldenness" of an artwork that the appropriate ratio is measured to avoid comparing apples and oranges. In general, a golden interval for the whole to greater ratio is narrower than the equivalent interval for the greater to lesser ratio.

G. Markowsky [5] suggests an acceptable interval from 1.58 to 1.66, which corresponds to a tolerance of just over 2%, although he does not stipulate to which type of ratio this interval applies when measuring line cuts. He offers 2% as a threshold after pointing out the magnifying effect of measurement errors on ratio calculations and that acceptable tolerances for structural engineering are 0.2%. I suggest that before we can decide on a golden interval, we need to have some idea about what change in ratio is discernible to the viewer.

There is a term from psychophysics called the *Just Noticeable Difference* (JND). JND is the smallest difference that is discernible between two empirically measurable stimuli. For example, in psychoacoustics, the JND of pitch is the smallest frequency change that is discernible to a listener. JND is applied to a variety of sensory perceptions, and a quick online search will find JND applications as varied as measuring discernible changes in speech-to-noise ratios, 3D depth perception, even the pungency of chilli peppers in a solution. JND is subjective and can be influenced by a variety of factors: for example, the JND for pitch is influenced by register, timbre and loudness of the pitches (see [4] pg. 162). For the purpose of finding a golden interval, we need to measure JND for changes in ratio. For our discussion, let us call a threshold at which a ratio change becomes discernible a *Ratio Just Noticeable Difference* or RJND.

Across a variety of sensory perceptions, JND has been found to conform roughly to Weber's Law (see [4] pg. 160-161) which states that JND is proportional to the intensity of the stimulus; in our case, this "intensity" is the size of the ratio. While there is no guarantee that perception of ratio is governed by Weber's Law, it seems obvious that it would be much more difficult to distinguish between ratios of 10 and 10.05 than between ratios of 1 (representing an equal cut or a square) and 1.05. In fact, when comparing rectangular ratios, research suggests that we are predisposed to perceive changes in the angle between the rectangle's diagonal and one of its sides rather than a change in ratio (see discussion regarding angles and line slopes in Cleveland and McGill [1]). Differentials give a small change in angle $\Delta\theta$ from a small change in rectangular ratio Δr as $\Delta\theta \approx \Delta r/(1+r^2)$. Thus, we cannot expect an RJND in the vicinity of $r = \phi$ to be the same as an RJND in the vicinity of some other ratio value. For an RJND to be relevant to our discussion, the ratios we test must be in the vicinity of ϕ.

Finding an RJND

There are various ways to design an experiment to determine an RJND. Firstly, there are different methods for finding a JND for any type of stimulus. In all cases, subjects are asked to compare two stimuli. In one version of the experiment, they are asked if the stimuli are the same or different, in another, they are asked which one is greater. As we discuss how to determine an RJND, we will refer to the version in which subjects are asked which ratio is greater, but the reader should keep in mind that either version is acceptable.

Next, subjects may be given the stimuli consecutively or simultaneously (side-by-side for visual stimuli). They may be permitted to view the stimuli repeatedly or only once for a limited amount of time. Because human perception is variable and subjective, consistent and fixed thresholds for noticing a change in a particular type of stimulus do not exist. Typically, a JND is determined by repeated comparisons of small changes and isolating the threshold change for which the subject answers correctly 75% of the time (note that 50% would be the expected result for randomly given answers).

Further variations of the experiment can be made for determining an RJND for a base ratio of ϕ. Subjects could be asked to compare ratios of ϕ and $\phi + \varepsilon$ for values of $\varepsilon > 0$. The threshold value may differ on either side of ϕ, requiring comparisons between ratios of ϕ and $\phi - \varepsilon$ as well. Alternatively, subjects could be asked to choose between ratios of $\phi - \varepsilon/2$ and $\phi + \varepsilon/2$. They could be asked to compare cuts in lines or rectangles. The lines or rectangles could be oriented horizontally or vertically.

The design of the experiment should be commensurate with the nature of the art that we intend to test. For example, if we are testing the earlier claim about the placement of eyes in a portrait, the RJND should be derived from a cut vertical line. On the other hand, if we are testing a claim about canvas proportions, then an RJND derived from rectangles would be more appropriate.

It should also emulate the spirit in which most of us view art, meaning that subjects should be prevented from carefully measuring lengths for the purpose of numerical comparison or computation. To frustrate any calculations of this sort, the line cuts and the rectangles compared should not have the same size (in the case of rectangles this means neither the same length nor width). Subjects should be unable to move the lines or rectangles for the purpose of aligning them. Otherwise, they may resort to comparing something other than the appearance of the ratios (see Cleveland and McGill [1]).

As with other JND measures, no doubt the value of an RJND will vary for different subjects; I would expect on average that visual artists have a keener eye than members of the general public.

Finding Golden Intervals for Line Cuts

Once we have determined an appropriate RJND, we can use it to help find a golden interval, *G*. We don't expect *G* to be unique: as we've discussed, the RJND we obtain from different experiments may vary; and, as we are about to explain, how we obtain a golden interval from an RJND depends not only on what we measure, but also our theory about the underlying reasons why ϕ might be aesthetically preferred.

Psychologist C. Green, in his survey [2] of studies of the golden ratio as an aesthetic preference, includes speculations by numerous researchers as to why a preference for ϕ might exist. These reasons range from mathematical, to biological, to cultural, even to correlating individual preferences to the proportions of the individual's visual field. How we obtain a golden interval from an RJND depends on which of these theories we use as a hypothesis.

For example, if we hypothesize that aesthetic preference for ϕ stems from cultural exposure, then it may be appropriate to define a golden interval *G* using $\phi \pm$ RJND (assuming RJND is found to be the same above and below ϕ). Note that this will give a golden interval for either the greater to lesser ratio, or the whole to greater ratio, according to which one was used in the experiment to establish the RJND. Unless directed otherwise, it seems likely that subjects will compare greater to lesser ratios.

On the other hand, if we hypothesize that aesthetic preference for ϕ stems from its mathematical definition, then I argue for a more subtle approach that focuses on the spirit of ϕ, rather than its numerical value. Kapraff [3] gives a possible reason for a preference based on the mathematical definition. He states that one of the key principles of good design, as originally proposed by the Roman architect Vetruvius, is the repetition of a small number of key ratios.

Recall that if a line is cut in the ratio ϕ, the ratio of greater to lesser equals the ratio of whole to greater. Thus, if a golden ratio cut is preferred because the viewer sees the same ratio twice, then any line cut for which the viewer cannot distinguish the two ratios, to all intents and purposes would appear to be golden and elicit a similar aesthetic response.

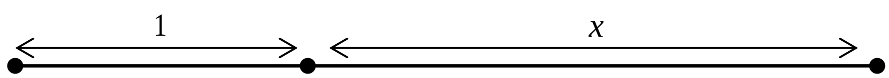
Thus, I propose another way of finding a golden interval for line cuts when our hypothesis is that the appeal of the golden ratio is the repetition of the ratio. We can adapt the RJND experiment specifically to suit the mathematical definition of the golden ratio. Golden intervals can be determined directly by showing single line cuts and asking subjects to compare the ratio of greater to lesser with the ratio of whole to greater. Note that subjects will need to have adequate numeracy skills to understand the question. The boundary of the golden interval (given in either greater to lesser or whole to greater form) would then be those ratios above and below ϕ for which subjects correctly identify the larger ratio 75% of the time.

In the absence of a golden interval from such an experiment, the next best approach is to use an RJND obtained from the more conventional experiments described earlier. We make the simplifying assumption that the threshold value for identifying the larger ratio in comparisons of greater to lesser and whole to greater ratios in a single line cut will be the same as an RJND from distinguishing ratios of greater to lesser in two line cuts.

Working from this assumption, let x be the ratio of greater to lesser and y the corresponding ratio of whole to greater. A ratio r (note that r might be x or y) is within the golden interval if and only if

$$|x - y| \leq RJND \qquad (1)$$

For simplicity we construct a line for which the greater is x and the lesser is 1.



Thus, the corresponding ratio of the whole to greater is $y = (x+1)/x$. From this we conclude that the golden interval for the ratio of greater to lesser is the set of solutions of:

$$\left| x - \frac{x+1}{x} \right| \leq z$$

where z is the value of the RJND. Solving $x - (x+1)/x = \pm z$ gives

$$x^2 - (1 \pm z)x - 1 = 0$$

$$x = \frac{1 \pm z \pm \sqrt{(1 \pm z)^2 + 4}}{2}$$

For $z > 0$, only two of the four values are positive, one from each quadratic equation:

$$x_1 = \frac{1 - z + \sqrt{(1-z)^2 + 4}}{2} \quad \text{and} \quad x_2 = \frac{1 + z + \sqrt{(1+z)^2 + 4}}{2}$$

These two values give a golden interval of $G = [x_1, x_2]$. Because we have solved inequality (1) for x rather than y, G is a golden interval for the greater to lesser ratio.

Table 1 gives golden intervals for the greater to lesser ratio, corresponding to an RJND of 0.1 and 0.05. Note that it may be appropriate to pad these slightly to account for measurement error. These intervals can be converted into golden intervals for whole to greater ratios by applying the conversion $y = (x + 1)/x$ to the interval endpoints.

Value of RJND	$z = 0.1$	$z = 0.05$
Golden Interval	[1.547, 1.691]	[1.582, 1.654]

Table 1: *Golden intervals for greater to lesser ratios.*

Finding Golden Intervals for Rectangular Proportions

Often the golden ratio is attributed to a work of art or architecture because the canvas of a picture or the outline of a building is close to a golden rectangle. Under these circumstances, how should we determine a golden interval? Once again, if our hypothesis is that a preference for ϕ is a consequence of something other than its mathematical definition, then we define the golden interval as $\phi \pm$ RJND where the RJND has been derived from an appropriate experiment involving rectangular proportions.

How do we proceed if we hypothesize that the aesthetic preference stems from the mathematical definition? The most likely aesthetic consequence of the mathematical definition is the following well-known construction in a golden rectangle. Any rectangle that is not square can be partitioned into two rectangles by adding a line segment perpendicular to its length so that one of the two smaller rectangles has the same proportions as the original rectangle. This rectangle is called a *reciprocal rectangle*, and the remaining rectangle is called a *gnomon*. If the original rectangle is golden, then the gnomon is a square as shown in Figure 1. Note that this construction can be executed (a) by beginning with the larger rectangle, as we have described, or (b) by beginning with the smaller golden rectangle and appending a square on the outside along one of its lengths.

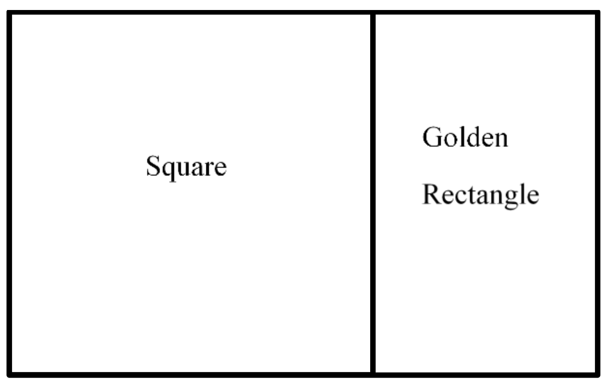


Figure 1: *Golden Rectangle Construction.*

It seems far-fetched to think that people consciously or subconsciously visualize either version of this construction when viewing a rectangle unless someone has shown it to them. But if an artwork includes the inscribed square, then we have more convincing evidence that the mathematical definition is at play.

Once again, we can adapt the RJND experiment to test this construction directly. Rather than presenting subjects with two separate rectangles to compare, subjects can be given a single rectangle with a partition that includes a square and asked to compare the proportions of the larger rectangle and the smaller rectangle. The boundaries of the golden interval are the rectangular ratios for which subjects correctly identify the rectangle with the greater ratio 75% of the time. The dimensions of the outer rectangle are analogous to the whole to greater ratio, and the dimensions of the inner rectangle give the corresponding greater to lesser ratio. So as with line cuts, it is important to record the ratios consistently, either from the larger rectangle or from the smaller, so that when the resulting golden interval is used to test a ratio from an artwork, it is clear which of the two rectangles must be measured.

Summary

Before we can decide if works of art have golden properties, we need to determine what ratio difference is discernible to a viewer. When selecting amongst the many proposed experiments for calculating an RJND, some basic principles must be followed. Firstly, the design of the experiment should be tailored to mimic the context in which the ratios are appearing in the works of art. Also, if we hypothesize that the golden ratio is aesthetically preferred because of its mathematical definition, we should design an experiment that will measure the golden interval directly based on the mathematical definition, or we should modify how we obtain a golden interval from an RJND obtained from more conventional tests. Finally, because measurements of the ratio of whole to greater and the ratio of greater to lesser must be treated differently, a golden interval for a line cut must be ratio specific so that the matching ratio from the artwork is measured before determining whether or not it is contained in the interval. This applies to rectangular ratios as well in cases where the measurement of the larger rectangle and the smaller (nearly) reciprocal rectangle are present and may be confusedly interchanged.

References

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