Dichromatic Dances

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Abstract

Dancers and choreographers employ symmetric patterns of bodies on stage fluidly and in fleeting ways. Ballet, folk, and contemporary dances often arrange dancers in two categories: men and women, two colors of costumes, antagonistic groups, etc. This paper investigates danced two-colored or dichromatic symmetry patterns, and continues an earlier investigation on how such danced symmetry patterns may be seamlessly morphed from one symmetry type to another, in a manner similar to visual parquet deformations. It also looks briefly at how they are employed in a variety of dance forms. These explorations may make for engaging activities for math classes or clubs.

Introduction

In a 2011 reprint of his 1964 work *Dance and Rhythm Mathematics* [14], Joseph Thie delineated several possible reasons for applying mathematical analysis to rhythm and dance: "One purpose is to show the mathematical aspects of rhythms and dances in order to give the creators and performers a deeper insight into their works. Another aim is to indicate the possible usefulness in the exploitation of mathematical techniques in musical rhythms and in the dance." Another reason given by Thie is it that it might give insight into how artists and mathematicians tend to communicate. We might add that it can also lead to interesting new mathematical problems. In this paper, I extend to two-colored or "dichromatic" patterns an exploration into danced parquet deformations [9], in which symmetric patterns of dancers morph from one pattern to another without breaking symmetry.

Symmetry can be a two-edged sword in choreography as in other art forms: too much symmetry has been decried as "lifeless" by the groundbreaking American choreographer Doris Humphrey [6], while too much asymmetry can be confusing and tiresome for audience members. Philip Ball in his book *Patterns in Nature* [1] looks at symmetry not from the point of view of trying to manipulate it in artistic works but at how patterns in nature "…arise from the breaking of perfect symmetry, that is from the disintegration of complete, boring uniformity where everything looks the same, everywhere…. The more symmetry that gets broken, the more subtle and elaborate the pattern." But choreographers often use symmetry in astute and well-crafted ways, as Robert Wechsler described in a 1990 article in Contact Quarterly [15],

> "In contemporary choreography we usually find symmetry playing a game of cat and mouse with us. Patterns form, change and are gone sometimes before we know what we have seen. When they first appear, symmetrical patterns catch our eye - there is a strong intuitive response. We recognize at once that something orderly has occurred, even if we have no precise idea what it was."

In this paper, I will look at three varieties of two-colored symmetry in dance and choreography: patterns of dancers rotating around a common center or rotocenter, frieze patterns of dancers in a straight line, and wallpaper patterns of dancers in symmetric formations in the plane. Sometimes these patterns overlap, for example in circular folk dance forms dancers often perform sequences of patterns around the circle's circumference that may be analyzed as linear frieze patterns bent around the circle. The human body is capable of a number of symmetries, including those shown in Figure 1(a). Other symmetries include helical symmetry, as in the normal swing of arms and legs while walking, or symmetries in time. Time symmetries include canon which is a translation in time, inversion which is the reversal of a sequence of movements as when hop then leap becomes leap then hop, and retrograde in which a movement is performed as if reversed in time (often difficult, since the body mechanics may be quite different).

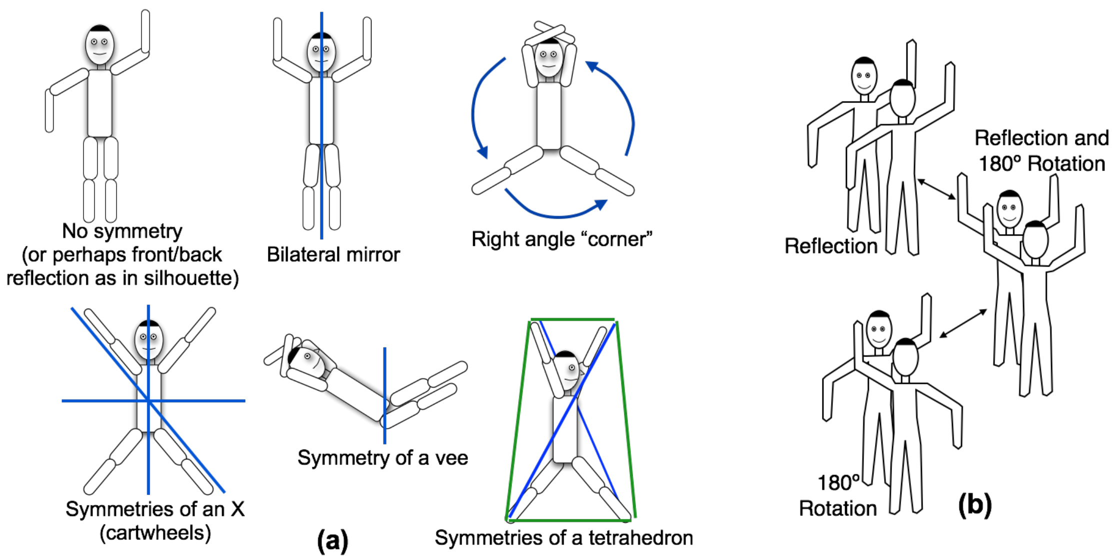

Figure 1: (a) *Various symmetries possible within one body.* (b) *Morphing between reflection and 180° rotational symmetries*

The simplest deformations from one symmetry to another may occur when dancers in a formation all maintain their vertical facings while moving fluidly to positions in which each dancer makes the same bilaterally symmetric shape, as shown in Figure 1(b). Here two dancers move between reflection and 180° rotation symmetry by shifting their arms to form an intermediate position in which they are in both reflection and rotational symmetry with each other. In this paper, we will assume all dancers are in vertical standing positions and maintain their facings as they make such shifts to individual bilateral symmetry. The intermediate bilateral formation will not lose any symmetries and may have gained new ones. The earlier paper [9] examined how this may allow parquet like deformations from one symmetric dance formation to another. In this paper we add the aspect of two colors: we assume the dancers are of two types, perhaps men and women, or half wearing white costumes, half wearing black costumes, for example.

Schwarzenberger [13] described how we may distinguish colored plane patterns according to two symmetry groups, G composed of symmetries which ignore the colors, and a subgroup H of those which preserve each of the colors. In the case of two-colored patterns the subgroup must be of index two, and Coxeter [4] uses these to establish notation of the form G/H to indicate each possible two-colored symmetry pattern. We will refer to this notation in this paper.

Dances with Central Symmetry

Figure 2 shows arrangements of four dancers in a square, in which the letters p, q, b, and d stand for dancers with right or left arms extended as seen from above, and the letters E stand for dancers in individual bilateral symmetry position. Figure 2(a) indicates the eight possible symmetries of the square, known as the dihedral group of order 8, often symbolized as D_4. We will assume these are also the possible symmetries for four dancers in a square formation: rotations of 0, 90, 180 or 270 degrees, and reflections through horizontal H, vertical V, or diagonal lines D_1 and D_2. In Figure 2(a) the two black letters are opposite each other, similarly for white; alternatively, the two black letters might be opposite white. Symmetries which invert colors are followed by a prime symbol. Note for example, that the four symmetries rotation by 0° or 180°, and reflection through H or V constitute a subgroup, known as the Klein four group, of index two in D_4. The designs in Figure 2(b) through 2(f) show examples of possible arrangements of four dancers with more than the trivial identity symmetry if only one color is allowed. Most compilations of 4-fold central symmetries include only the non-colored version of Figure 2(a) and Figure 2(d). We may modify these non-colored arrangements by assigning two letters to be black and two white, in such a way that the arrangement has a subset of the symmetries of D_4. The 2-color possibilities were further delineated by Shubnikov [12]. In the following we use some symmetries counted as 2-fold by some writers, such as Figure 2(b), (c), (e), and (f), as they are easily displayed by 4 dancers, but do not use all of Shubnikov's designs, as some of those would result in our dancers being upside down!

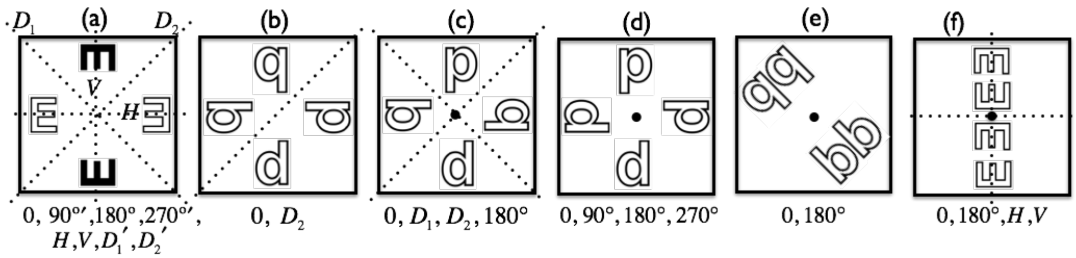

Figure 2: *Arrangements of four dancers in a square, with the symmetries listed.*

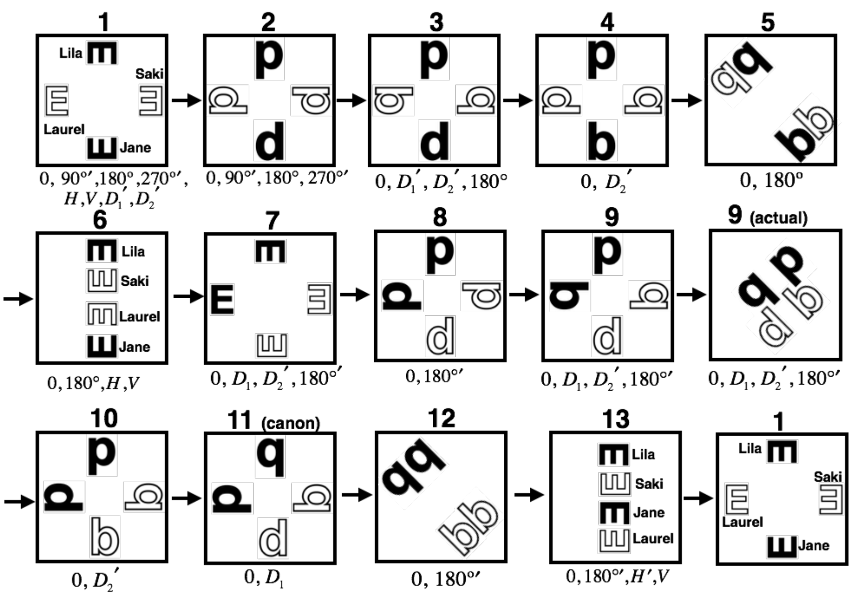

Figure 3: *Diagram for 2017 dance "Blacks and Whites."*

Figure 3 shows a "parquet deformation" sequence of positions for 4 dancers from a recent dance by the author titled "Blacks and Whites," using possible two colorings of the one-color designs from Figure 2. The transitions between positions shown are all either position 1 or 7. Positions 4, 10, and 11, for example, are based on Figure 2(b), and positions 3 and 9 on Figure 2(c). A recent rehearsal of this sequence is shown in [10]. Each position includes its list of symmetries; note, for example, that 4 and 10 have the same symmetries but are different patterns [11].

As in almost any dance, positions in "Blacks and Whites" are fluid and not simply visualizations of rigid symmetry positions. For example, position 9 moves quickly into a diagonal pattern labeled "9 (actual)", and position 11 is performed as a canon, as suggested by the dancers in order to break the predictability of the dance. Another section of the dance creates a context for the symmetry positions as commentary on how we see black and white in a political and cultural context. And the dance was also part of a show

Choreocopia, a celebration of food, song, and dance in which black and white culinary items played a role. As a choreographer, I strenuously reject the idea that works of art containing mathematical elements must primarily be representations of mathematical theorems or concepts, but instead see mathematics as part of the palette with which we may paint movement designs. The process of using danced symmetry deformations might yet be of interest to a math class or club exploring symmetries of dance; such a class might divide into groups of 4 with each group tasked with creating their own sequence of movements using a similar sequence of designs, and might also explore how to relate the dance to other ideas or elements.

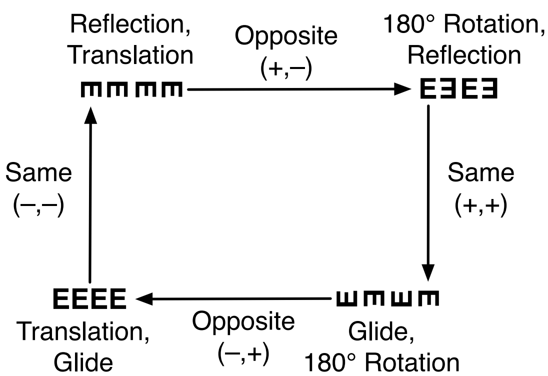

Figure 4: *"Frieze-cycle" for bilaterally symmetric dancers in linear arrangements.*

Frieze symmetries

Figure 4 is similar to diagrams in [9], showing symmetric deformations between linear arrangements of bilaterally symmetric dancers. Mathematically we assume the lines of dancers extend infinitely in both directions, though in performance lines of dancers may start at stage left and "disappear" offstage right, or may be arranged in a circle as is common in many folk dance forms. Note that the sequences of Es in translation symmetry are opposite the sequences in rotation symmetry, and the sequences of Es in mirror symmetry are opposite the glides; a translation may be considered to be a rotation with an infinite radius and a reflection may be considered to be a glide reflection with zero translation. The notation (–,–) along the arrow leaving EEEE indicates that the deformation transition involves the Es in both odd positions and in even positions in the sequence of Es rotating together –90°, the deformation transition along the arrow leading to EEEE involves the Es in odd positions rotating –90° and the Es in even positions rotating +90°, and similarly for the other transitions. Here –90° means the dancer rotates to the new position 90° in the standard negative direction.

There are 17 dichromatic frieze patterns [5], shown in Figure 5, with each connected to the associated pattern produced when the dancers quickly shift to positions with bilateral symmetry. All are shown in two rows, to unify the presentation, even though those without a horizontal mirror need only

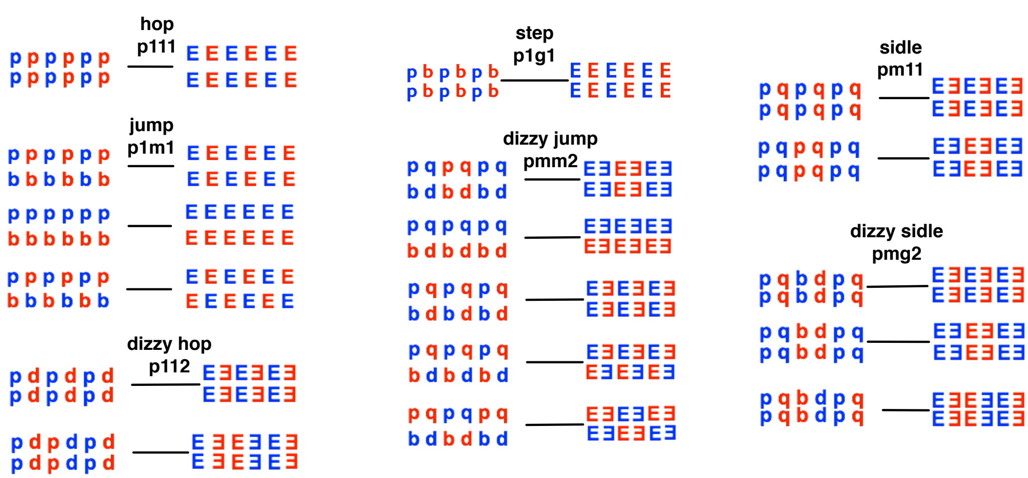

Figure 5: The 17 *dichromatic frieze patterns, based on 7 monochromatic patterns, each showing associated bilateral pattern, with Conway and IUC titles.*

one row to demonstrate the pattern. Figure 6 shows how eight of the 17 two-colored patterns may be produced by the cycle shown in Figure 4, in this case using only one row of dancers for each pattern.

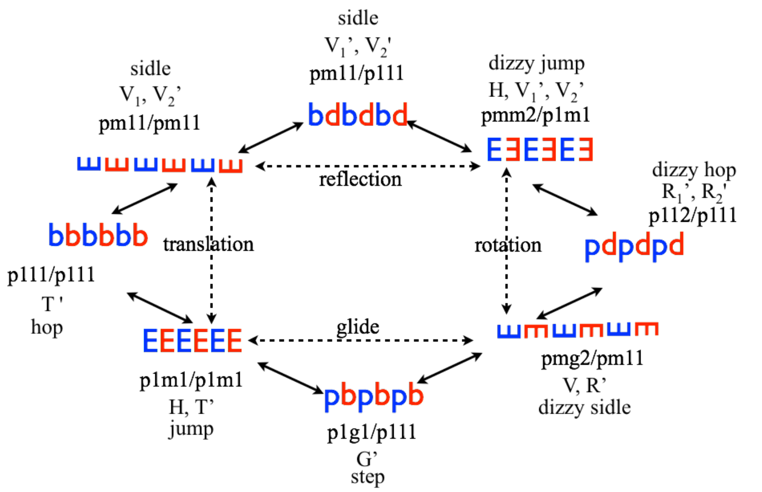

Figure 6: *Eight of the 17 dichromatic frieze patterns produced by the cycle shown in Figure 4.*

Also indicated in Figure 6 are John Conway's colorful titles of the underlying monochromatic pattern, Coxeter's use of the International Union of Crystallography (IUC) dual notation for the dichromatic pattern, and letter symbols representing generators of the symmetry group for the dichromatic pattern. For example, the symmetries of the sidle pattern of sideways Es in the upper left, with signature pm11/pm11 is generated by two distinct vertical reflections, indicated by V_1 and V_2'. V_1 represents a reflection line through the center of one of the Es which preserves all colors, and V_2' represents a vertical reflection line between Es which switches all colors. The letters T, R, and G represent translations, half-turn rotations, and glides respectively. The dotted arrows in Figure 6 represent the symmetries common to the three patterns at the ends of the arrows and between the ends, similar to Figure 4. The other nine of the 17 dichromatic frieze patterns do not arise as conveniently from the cycle pattern.

We can connect all 17 of the dichromatic patterns using bilateral symmetry as in Figure 7, though this diagram does not have the overall logic found with the seven monochromatic frieze patterns. In this

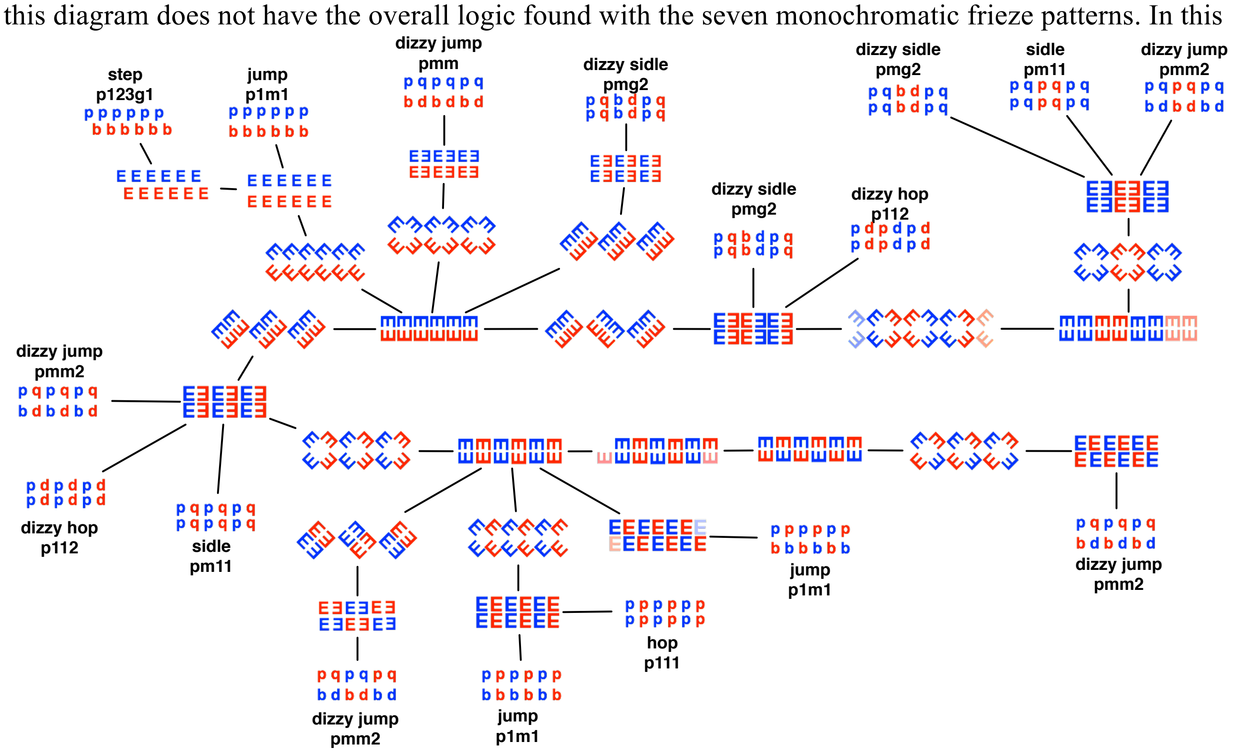

Figure 7: The 17 *dichromatic frieze patterns are the leaves of this graph.*

diagram the 17 patterns are the leaves of the graph and the intermediate nodes indicate the directions in which the dancers turn to move to the next node.

Figure 8 shows two examples of frieze deformations in two forms of dance. The upper display is from the section 0:24 – 1:15 of the Waltz of the Flowers, performed by the Moscow Ballet in 2014 [7]. Figure 8 shows the stage right chorus dancers, who are mirrored on stage left by a similar group of dancers. The lower display is from the section 0:28 – 1:10 of the Circle Waltz performed by the Central Illinois Civil War Dance Society in 2011 [2].

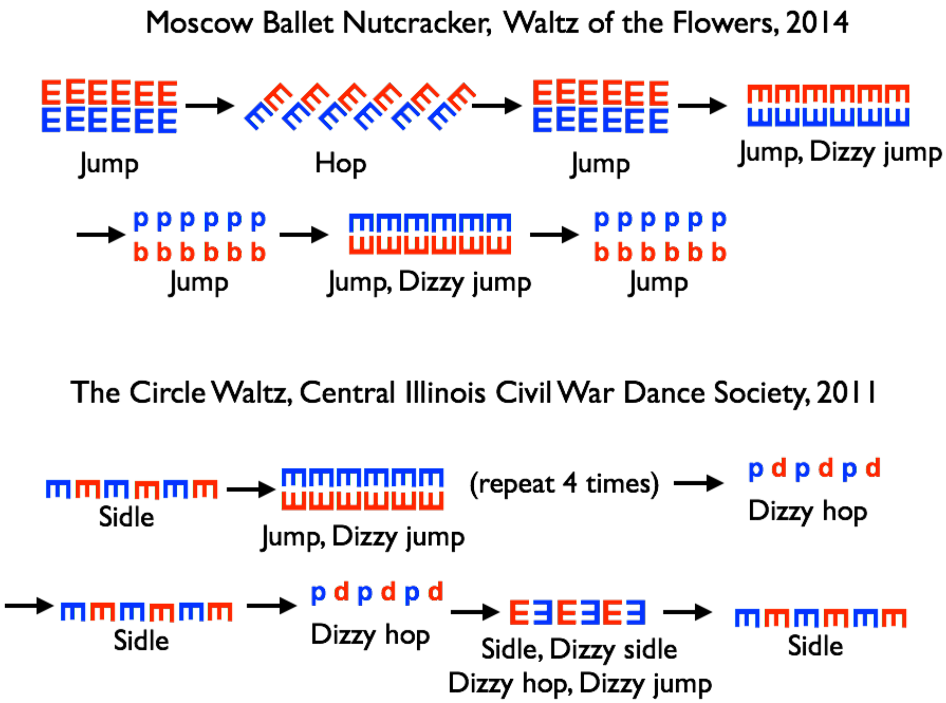

Figure 8: *Two examples of frieze deformations.*

Wallpaper Symmetries

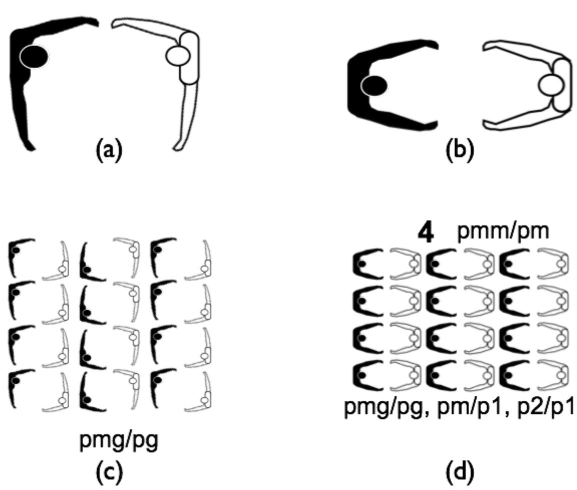

Figure 9: *Wallpaper pattern examples.*

There are 46 two-colored wallpaper symmetry patterns [5]. In this paper, we delineate them and show the associated bilateral pattern, but have not listed all possible deformations between them or given dance examples of these deformations. Figure 9(a) shows designs for black or white costumed dancers as seen from above, each dancer in an asymmetrical stance. Figure 9(b) shows the same dancers in bilateral symmetry, also seen from above. Figure 9(c) shows one of the two-colored wallpaper patterns of dancers in pmg/pg formation, using the IUC designations for the two associated symmetry groups, as delineated by Coxeter. Figure 9(d) is what the design in Figure 9c would look like if the dancers move to bilaterally symmetric positions. The number 4 indicates that it is the 4th in the list of 30 such formations. The designation pmm/pm above Figure 9(d) is the IUC dichromatic designation for the symmetries of the bilateral pattern. Below Figure 9(d) are three dichromatic wallpaper patterns that reduce to Figure 9(d) when the dancers move to bilaterally symmetric positions. The full set of both the 46 two-colored wallpaper patterns and the 30 associated bilateral designs are shown in [8], using figures and symbols like those shown in Figure 9(c) and (d). The table in Figure 10 lists several notations used for dichromatic wallpaper patterns. The first column shows the notation used by Grunbaum and Shephard [5], the second column shows the IUC dichromatic symbols, the third column shows the pair of symmetry groups listed by Coxeter [4], and the fourth column shows the orbifold notation as in Conway et al [3]. The fifth column shows the Coxeter notation for the formation when the dancers move to bilaterally symmetric positions, and the sixth column gives the number 1 through 30 of the formation found in the accompanying set of diagrams [8], such as shown in the example in Figure 9(d).

A primary task undertaken in this paper has been the examination of ways of morphing one symmetric dichromatic pattern to another using intermediate bilaterally symmetric body formations, and the compilation of the set of such designs based on known dichromatic symmetry patterns. This paper and the one preceding it in 2014 show that much more could be explored in terms of how dancers and choreographers make use of symmetry. Thank you to Darrah Chavey for suggesting the exploration of color patterns in choreography; he has also suggested further work examining symmetries in time as they relate to spatial symmetries between dancers.

Grunb./ Shep.	**Int. (IUC)**	**Coxeter**	**Orbif.**	**Bilateral**	**#**
p1[2]	p'$_b$1	p1/p1	o/o	pm/cm	1
pg[2]$_1$	pg'	pg/p1	xx/o	pg/pm	2
pg[2]$_2$	p'$_b$1g	pg/pg	xx/xx	pg//g	3
pm[2]$_1$	p'$_b$g	pm/pg	**/xx	pmm/pmg	14
pm[2]$_2$	c'm	pm/cm	**/*x	pmm/cmm	5
pm[2]$_3$	p'$_b$m	pm/pm(m)	**/**(1)	pmm/pmm	6
pm[2]$_4$	pm'	pm/p1	**/o	pmm/pm	4
pm[2]$_5$	p'$_b$1m	pm/pm(m’)	**/**(2)	pmm/pmm	7
cm[2]$_1$	cm’	cm/p1	*x/o	pmg/pg	8
cm[2]$_2$	p’$_c$g	cm/pg	*x/xx	pmg/pgg	9
cm[2]$_3$	p’$_c$m	cm/pm	*x/**	pmg/pmm	10
p2[2]$_1$	p2’	p2/p1	2222/o	pmm/pm	4
p2[2]$_2$	p’$_b$2	p2/p2	2222/2222	pmm/pmm	7
pgg[2]$_1$	pgg’	pgg/pg	22x/xx	pmg/pgg	9
pgg[2]$_2$	pg’g’	pgg/p2	22x/2222	pmg/pmm	10
pmg[2]$_1$	p’$_b$mg	pmg/pmg	22*/22*	pmm/pmm	6
pmg[2]$_2$	pm’g	pmg/pg	22*/xx	pmm/pm	4
pmg[2]$_3$	p’$_b$gg	pmg/pgg	22*/22x	pmm/pmg	14
pmg[2]$_4$	pmg’	pmg/pm	22*/**	pmm/pmm	7
pmg[2]$_5$	pm’g’	pmg/p2	22*/2222	pmm/pmm	7
pmm[2]$_1$	p’$_b$mm	pmm/pmm	*2222/*2222	pmm/pmm	12
pmm[2]$_2$	pmm’	pmm/pm	*2222/**	pmm/pmm	6
pmm[2]$_3$	c’mm	pmm/cmm	*2222/2*22	pmm/cmm	13
pmm[2]$_4$	p’$_b$gm	pmm/pmg	*2222/22*	pmm/cmm	11
pmm[2]$_5$	pm’m’	pmm/p2	*2222/2222	pmm/pmg	14
cmm[2]$_1$	p’$_c$gg	cmm/pgg	2*22/22x	pmm/pmg	14
cmm[2]$_2$	cmm’	cmm/cm	2*22/*x	pmm/cmm	11

cmm[2]$_3$	p'$_c$mg	cmm/pmg	2*22/22*	pmm/pmm	6
cmm[2]$_4$	cm'm'	cmm/p2	2*22/2222	pmm/cmm	5
cmm[2]$_5$	p'$_c$mm	cmm/pmm	2*22/*2222	pmm/pmm	7
p31m[2]	p31m'	p31m/p3	3*3/333	p31m/p3	25
p3m1[2]	p3m'	p3m1/p3	*333/333	p3m1/p3	26
p4[2]$_1$	p'$_c$4	p4/p4	442/442	p4/p4	15
p4[2]$_2$	p4'	p4/p2	442/2222	p4/pmg	17
p4g[2]$_1$	p4g'm'	p4g/p4	4*2/4*2	p4g/p4	16
p4g[2]$_2$	p4'g'm	p4g/cmm	4*2/2*22	p4g/cmm	18
p4g[2]$_3$	p4'gm'	p4g/pgg	4*2/22x	p4g/p2	19
p4m[2]$_1$	p'$_c$4gm	p4m/p4g	*442/4*2	p4m/p4g	20
p4m[2]$_2$	p4m'm'	p4m/p4	*442/442	p4m/p4	21
p4m[2]$_3$	p4'm'm	p4m/cmm	*442/2*22	p4m/p2	22
p4m[2]$_4$	p4'mm'	p4m/pmm	*442/*2222	p4m/pmm	23
p4m[2]$_5$	p'$_c$4mm	p4m/p4m	*442/*442	p4m/p4m	24
p6[2]	p6'	p6/p3	632/333	p6/p3	27
p6m[2]$_1$	p6'm'm	p6m/p31m	*632/3*3	p6m/p31m	28
p6m[2]$_2$	p6'mm'	p6m/p3m1	*632/*333	p6m/p3m1	29
p6m[2]$_3$	p6m'm'	p6m/p6	*632/632	p6m/p3	30

Figure 10: *Notations for wallpaper patterns plus associated bilateral formations.*

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