The Artful Kaleidoscopes of the Circular and Spherical Bells

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Abstract

The Gaussian bell, over one or more dimensions, is one of the most ubiquitous mathematical objects in science. This work explains, essentially summarizing previously published research, how the iteration of simple maps leads to universal constructions of bells, over two and three dimensions, which surprisingly define vast assortments of exotic kaleidoscopic decompositions of the circular and spherical bells in terms of crystalline patterns that include, among others, the geometric structure of ice crystals and that of the DNA rosette.

Introduction

In order to set the stage, it is pertinent to start by explaining a mathematical game that uses two simple maps, from the plane to the plane:

$$w_1(x,y) = (0.5 \cdot x, x + d_1 \cdot y)$$
$$w_2(x,y) = \left(0.5 \cdot x + 0.5, 1 - x + d_2 \cdot y\right),$$

where d_1 and d_2 are parameters. As it may be shown that $w_1(0,0) = (0,0)$, $w_2(1,0) = (1,0)$ and $w_1(1,0) = w_2(0,0) = (0.5,1)$, it happens that w_1 operates to the *left* and w_2 to the *right* of the domain of the maps, which is the interval [0,1].

The game is one of chance and to play it one needs a coin. The tosses determine which map to use, say, w_1 if heads and w_2 if tails. Figure 1(a) plots the aforementioned three points and highlights the middle one (0.5,1) to start the game. Then, select $d_1 = -d_2 = 0.5$ and imagine that the first toss is a head, then using w_1 on the middle point yields a point to the *left*, as shown in Figure 1(b). Suppose now that a head occurs again, then such gives the point marked in Figure 1(c), If now there comes a first tail, the next point will be on the *right*, as in Figure 1(d). Figure 1(e) shows what is found when this *iterative* process is carried out 100 times, and Figure 1(f) portrays the eventual set obtained, which is known as the *attractor* of the maps.

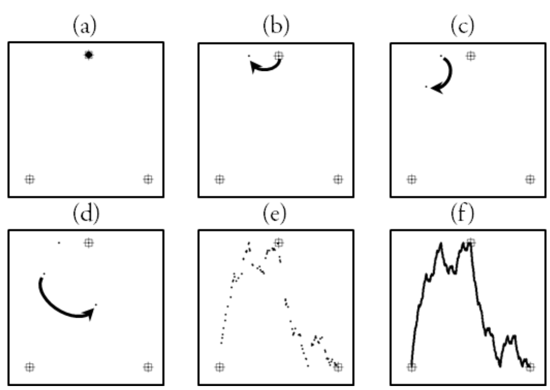

Figure 1: *Progressive random usage of two simple maps and their ultimate attractor.*

Following the seminal work of Barnsley [1], it may be shown that the random iteration of the simple maps herein always produces (irrespective of the coin tosses of a fair or a biased coin) the interesting geometric attractors in Figure 2, whose shapes vary depending on the signs of the parameters d_1 and d_2. As seen, such sets are shaped as convoluted *wires*, from x in the horizontal to y in the vertical, that pass through the aforementioned three points, and they evoke either mountain or cloud profiles. The $-+$ case is not shown as it mirrors the $+-$ case.

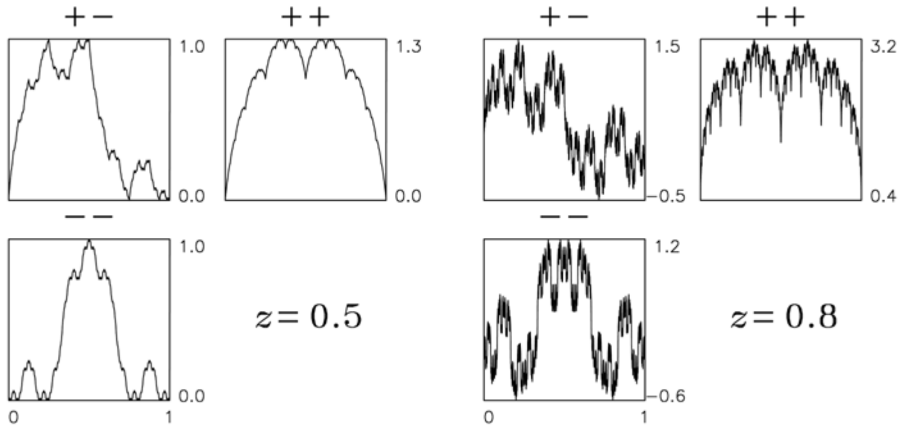

Figure 2: *Attractors of the maps herein for shown sign combinations of d_1 and d_2. $|d_1| = |d_2| = z$.*

As seen, as the magnitude of z increases beyond 0.5, the wires become thicker and hence require for their plots increasing amounts of ink. This means that beyond $z = 0.5$, such wires become *fractal* objects as they may be assigned non-integer dimensions greater than 1. In fact, when z tends to 1 (which is the maximum value that guarantees the existence of a connected attractor) the wires become so massive that they tend to fill the whole two dimensional space.

Shadows from Fractal Wires

It happens that playing the iterations game not only defines a unique attractor, but also a unique *texture* over such a wire that reflects how the attractor is filled. This means that using a fair or a biased coin will result in distinct textures. Using a biased 70-30 coin (one that implies using w_1, going left, 70% of the time and w_2, going right, 30% of the time) makes the acquired points be more to the left than to the right. As seen in Figure 3 and as introduced by Puente [3], such defines interesting stable projections (shadows) dx and dy, simply by computing histograms of sequential points over the x and y coordinates.

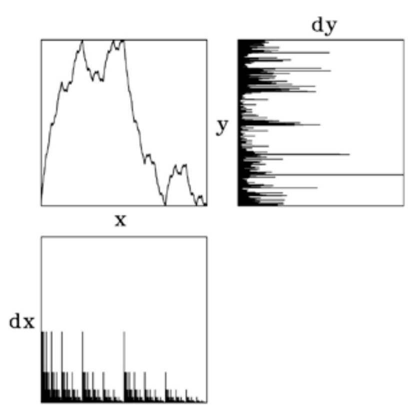

Figure 3: *Shadows dx and dy, over x and y, from iterations giving the wire mountain of Figure 2.*

The flow of the construction goes from x into y and the "output" dy may be thought of as the "shadow" made by the wire when "illuminated" by the "input" dx. As seen, dx has a decisive repetition, which turns out to reflect, in its precise layering, the so-called *multifractal* structure seen in fully developed turbulence, as discovered by Meneveau and Sreenivasan [2]. The texture dy, derived ultimately without chance from dx via the wire function, exhibits a seemingly random structure that defines, generalizing the maps and varying their parameters and given analogies of the notions with the famous allegory of the caveman, a Platonic approach to natural complexity, as introduced by Puente [9].

When the parameters $d_1 = -d_2 = z$ tend to the limit of one, the attracting wire tends to fill up two-dimensional space and the shadow dy approaches a Gaussian bell, not only for a multifractal input dx as illustrated in Figure 4 when $z = 0.999$, but also for any non-discrete input dx, which includes using coins having arbitrary biases. This result, proven by Puente et al. [5], establishes an unforeseen bridge between *disorder* and *order*, as the maximally infinite wire transforms the violent, and ultimately dissipative, spikes of turbulence into the harmonious and smooth bell associated with heat conduction.

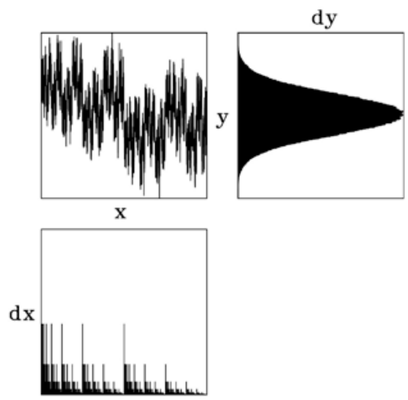

Figure 4: *From a multifractal to a Gaussian bell via a space-filling wire.*

Extensions to Higher Dimensions

The notions leading to wires and projections may be extended to higher dimensions so that the iteration of simple maps, but with more coordinates, produce attractors either from a line into a plane – from x into (y, z) – or from a line into a volume – from x into (y, z, w) – yielding wires having fractal properties as they have non-integer dimensions that now range from 1 to 3, or from 1 to 4, respectively.

In the three dimensional case and for wires passing by the points $\{(0,0,0), (0.5,1,1), (1,0,0)\}$, the construction entails iterating

$$w_1\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0.5 & 0 & 0\\1 & d_1 & h_1\\1 & l_1 & m_1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}, w_2\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0.5 & 0 & 0\\-1 & d_2 & h_2\\-1 & l_2 & m_2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}0.5\\1\\1\end{pmatrix},$$

in which the previous parameters d_1 and d_2 are substituted by matrices, which in polar coordinates are:

$$\begin{pmatrix}d_1 & h_1\\l_1 & m_1\end{pmatrix} = \begin{pmatrix}r_1^{(1)}\cos\theta_1^{(1)} & -r_1^{(2)} sin\,\theta_1^{(2)}\\r_1^{(1)}\sin\theta_1^{(1)} & r_1^{(2)}\cos\theta_1^{(2)}\end{pmatrix}, \begin{pmatrix}d_2 & h_2\\l_2 & m_2\end{pmatrix} = \begin{pmatrix}r_2^{(1)}\cos\theta_2^{(1)} & -r_2^{(2)} sin\,\theta_2^{(2)}\\r_2^{(1)}\sin\theta_2^{(1)} & r_2^{(2)}\cos\theta_2^{(2)}\end{pmatrix}.$$

It happens that the obtained output textures (histograms), now over the plane (y, z) and over the lines y and z, besides adding to the notion that iterations of simple maps may be used to define a Platonic approach to natural complexity, also result in limiting two-dimensional bells, which happen when the magnitudes of all radial parameters $r_n^{(j)}$ tend to one and when the involved angular parameters are synchronized, $\theta_n^{(1)} = \theta_n^{(2)} + k_n\,\pi$, for k_n integer.

There are sixteen cases on the sign combinations of $r_n^{(j)}$ values. Twelve of them happen to define Gaussian bells and the other four oscillations among several bells, Puente et al. [11]. The most common case, the one of a circular bell, is illustrated in Figure 5. Here, an input multifractal dx (the same as before and reflecting usage of a biased 70-30 coin and shown in the bottom center) gets transformed, by the thick wire from x to (y, z), shown projected from x to y and from x to z on the right, into a bivariate bell that yields dyz from above and dy and dz from the sides..

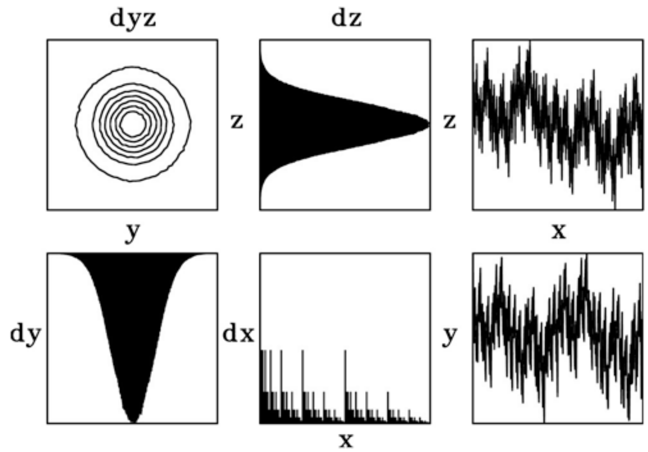

Figure 5: *From a multifractal dx to a circular two-dimensional Gaussian bell.*

As before, the obtained limiting textures happen universally except for discrete input illuminations. As explained by Puente and Klebanoff [4], while there is a proof for the aforementioned one dimensional bell in Figure 4, a similar demonstration for the two-dimensional case is not easily found, and such suggested studying how circles, as in Figure 5, are formed while performing the iterations.

Exotic Beauty in Circular Bells

As discovered by Puente [7], when successive iteration values are plotted over the (y, z) plane in successive groups of, say, 2,000 dots, surprises emerge. As illustrated in Figure 6, with successive shapes plotted by columns from top to bottom and from left to right, the circular bell turns out to be made of beautiful crystalline patterns, which turn out to reflect a graceful central limit theorem.

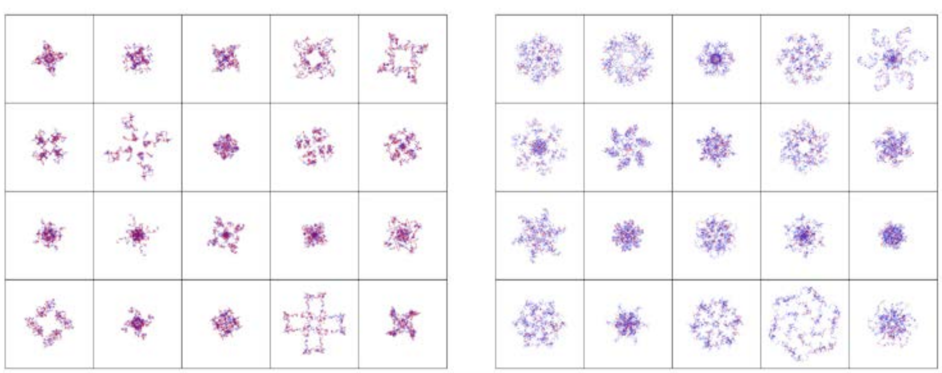

Figure 6: *Exotic kaleidoscopes inside the circular bell for 90° and 60° angular parameters $\theta_n^{(j)}$.*

The ever-changing patterns inside the bell make up unforeseen kaleidoscopes that evoke the concept of “the aleph,” as introduced by Jorge Luis Borges in his famous tale by the same name. Quite literally, selecting radial parameters with magnitudes close to 1 ($|r_n^{(j)}| \approx 1$) and angles dividing exactly 360° define a “point of light” (to quote Borges) from which to see great many shapes that makes explicit *hidden order in chance*.

As illustrated in Figure 7 for sample sets with 20,000 dots, the circular bell contains a myriad of patterns having arbitrary n-fold symmetries, interesting *rosettes*, such as the ones admired by several civilizations, that may be classified, depending on the signs of the radial parameters, as having radial or rotational traits, as reported by Puente [8].

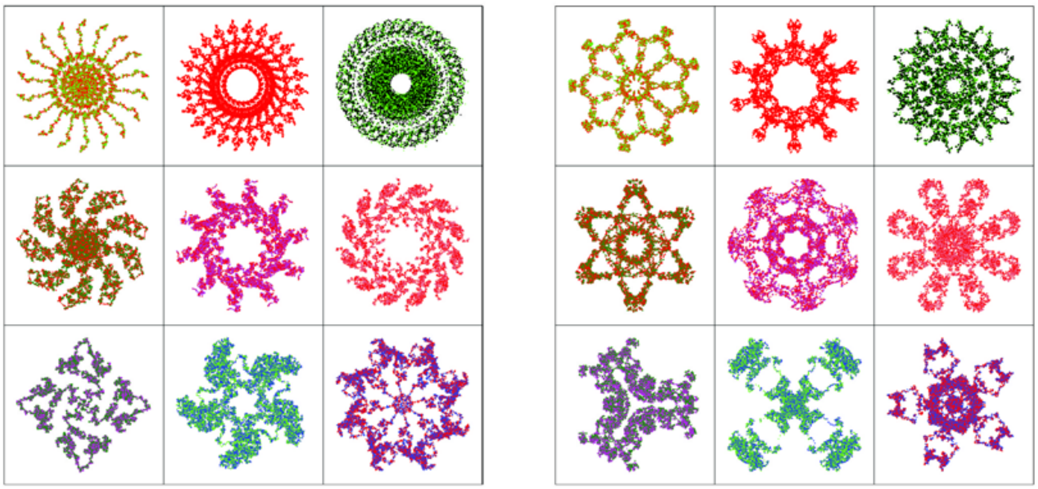

Figure 7: *Sample rotational and radial rosettes inside the circular bell.*

Just iterating simple maps, it may be shown that the geometric structure of nature's ice crystals are found inside circular bells as concealed mathematical designs. This is illustrated in Figure 8, which portrays nine crystals inside the bell that match photographed crystals, whose templates were successively filled pasting together pseudo-random sequences of iterations using a nearly space-filling wire having angular parameters equal to 60°. As explained by Puente [8], these sets are made of a variable number of points, which range from 83,000 to 164,000 dots.

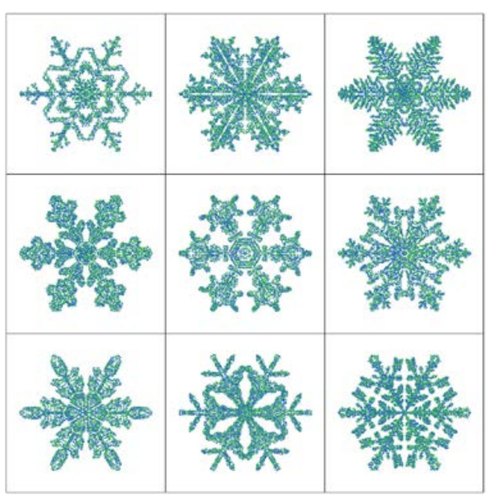

Figure 8: *Sample ice crystals inside the bell.*

The bell contains indeed a plethora of ice crystals, like the ones in Figure 9, which were obtained by Puente and Puente [10] performing the iterations of two simple maps having angular parameters equal to 60° and according to the binary expansion of π. These crystals, shaped as stars and sectors and made of 100,000 points, have specific shapes that exquisitely depend on the precise sequence of 0's and 1's employed. Being slices of the circular bell, these crystals grow by diffusion, as in nature, and, as they only appear in the limit when the corresponding wire fills up space (and not before), they may be thought to be born in the "*plenitude of dimension*," as coined by Puente [8].

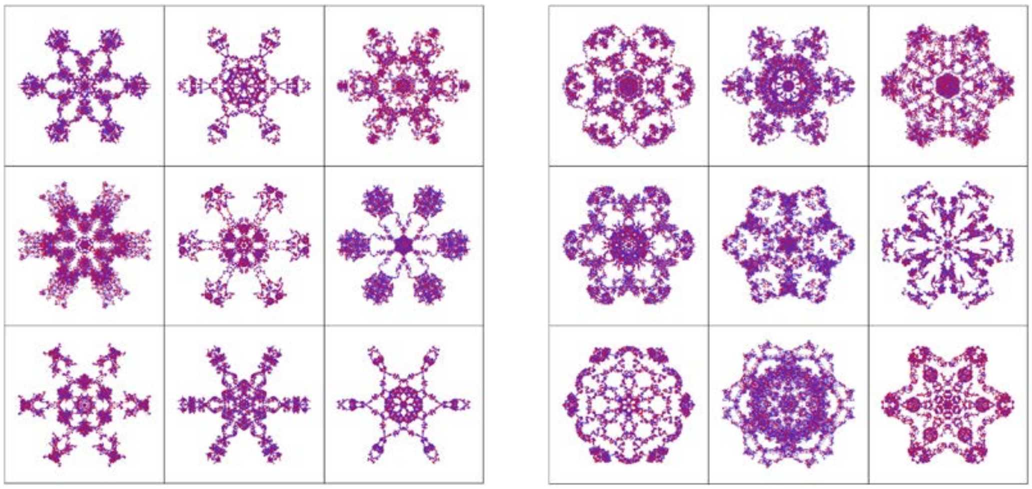

Figure 9: *Ice crystals inside the circular bell encoded via the binary expansion of* π.

A single infinite space-filling wire may surely encode infinitely many rosettes that may be made explicit, say, performing the iterations according to the binary expansions of irrational numbers. Although it may not be easily surmised if any irrational number will ultimately define the same patterns that, say, π does, space-filling attractors may certainly store vast amounts of information in a manner that boggles the mind.

There is yet, however, another striking application. The rosette of DNA, one having ten sided symmetry as the double helix twists base by base by 36°, is also found inside the Gaussian bell. Although there are several alternative iterations that may fit a template of the pattern associated with life (Figure 10 left, as it appears in biochemistry textbooks), it happens that such pattern is also nicely approximated while performing the iterations of two suitable maps using the binary expansion of π. In fact, choosing two maps with angular parameters equal to 36° and iterating them guided by the first 40,000 bits of π yields a rosette that, very improbably, resembles the natural one (Figure 10, right). As explained by Puente [7], this is a rather suggestive finding, given the prominence of π, which, in addition, hints at an organizational principle based on *geometric design*, rather than blind chance.

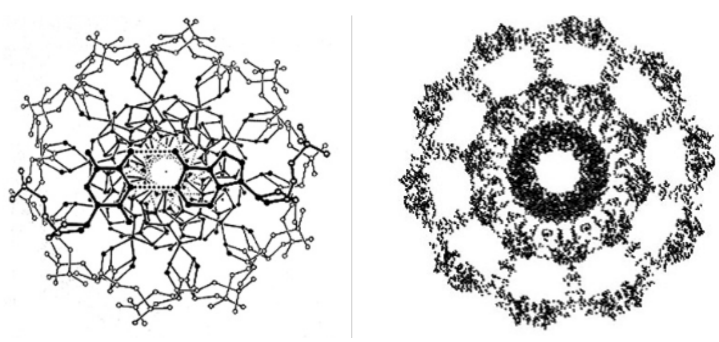

Figure 10: *The DNA rosette and a counterpart inside the bell coded by the binary expansion of* π.

Exotic Beauty in Spherical Bells

The construction of wires over four dimensions is similar and requires replacing the 2x2 matrices in polar coordinates by 3x3 matrices in spherical coordinates, as explained in Puente et al. [13]. Some of the sixty four cases on the sign combinations of the new radial parameters do result, as may be expected, in beautiful higher order kaleidoscopes, as illustrated over distinct spaces for three sets of 10,000 dots each in Figure 11.

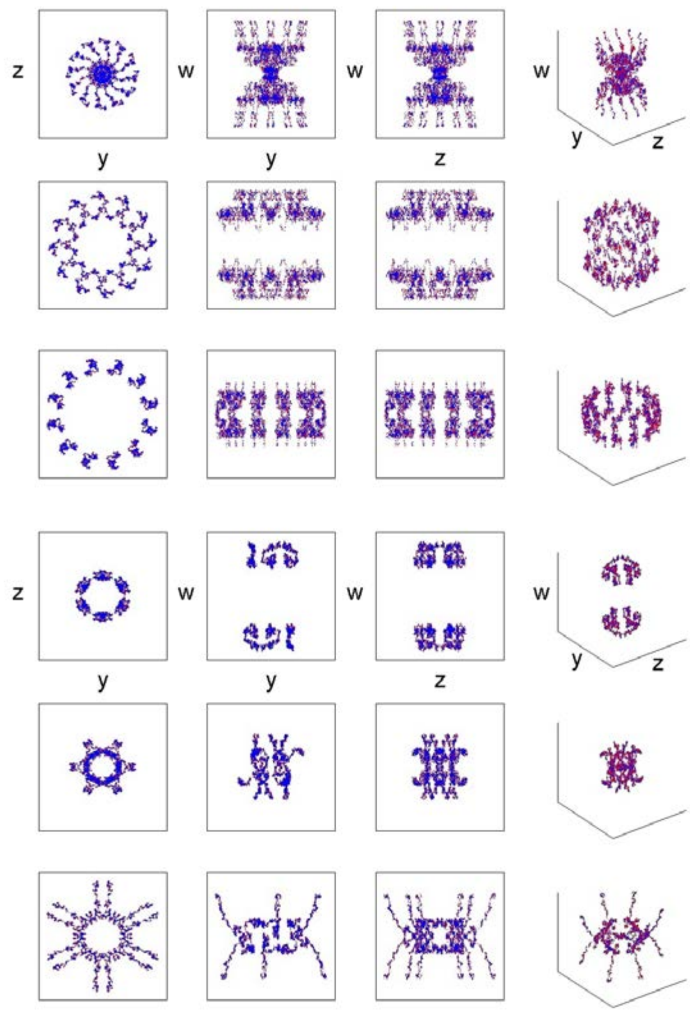

Figure 11: *Two cases of sample three-dimensional sequential patterns inside the spherical bell.*

Concluding Remarks

This work has made explicit hidden kaleidoscopic treasures inside circular and spherical Gaussian bells that would inspire wonderment the next time you listen to a bell. Despite all this beauty, there is a case worthy of further reflection that corresponds to a limiting wire, from x to y, which defines an amazing bell concentrated at infinity that happens to embody, by the shape of the bell and the associated statistical independence of an implied central limit theorem, true freedom. Such a case is the space-filling cloud of the plus–plus choice, which inspired me to write the following verses, as further explained in Puente [12].

The bell peals silent,
reflecting its peace,
and inside it gathers
lovely masterpiece.

Exotic pure beauty,
o splendid delight,
this limit in fullness
stores life's designs.

Such vessel contains,
alephs of all tastes,
diatoms and crystals
including DNA.

But there is a case,
reason to rejoice:
o forward selection
that raises it all.

O plus–plus election
that opens the door.

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