Sculptural Forms Based on Radially-developing Fractal Curves

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Abstract

Fractal curves that develop spatially in a linear manner over several generations give rise to captivating sculptural forms. A variety of such structures are explored using different fractal curves, with the use of multiple copies allowing the creation of closed sculptural forms that are more vase- or pot-like. More complex, visually rich, and natural forms are generated by developing fractal curves radially in three dimensions. Computer graphics, 3D printing, paper folding, and ceramic sculpture are used to explore and elaborate these constructions.

Introduction

Mathematics can serve as a tool to guide the creation of new sculptural forms that artists would probably not come up with otherwise. A case in point is the spatially-developing fractal curves described by Irving and Segerman [5]. In these structures, the first several generations of a fractal curve are separated in the direction orthogonal to the plane of the curve. Successive generations are then joined by polygons or *via* interpolation by curved surfaces, resulting in a fractal surface in the limit.

One specific example is the use of a curvilinear version of the terdragon curve (Fig. 4c), which served as the starting point for a ceramic sculpture entitled "Three-fold Development", shown in Figure 1a. Three copies of the curve were joined to form a closed curve, where the starting point (base of the sculpture) is a circle. A modified version of the Sierpinski curve [6] was used as the basis for the sculpture of Figure 1b, "Four-fold Development".

The surfaces described above resemble hyperbolic surfaces. Hyperbolic space has constant negative curvature. When a hyperbolic surface is embedded in Euclidean 3-space, every point on the surface is a saddle point. As the structures above progress through iterations of a fractal curve a cross section of the surface increases in length.

Related fractal hyperbolic surfaces have been created by using fractal tilings embedded in 3-space [3]. Fathauer has also demonstrated that certain developing fractal curves can be created by folding simple Pythagorean trees [4]. In both these cases the surfaces are not strictly hyperbolic, as the individual tiles have zero curvature.

The sculptures in Figure 1 develop linearly, so that the bottom and top curves bounding the surface lie in parallel planes, as illustrated in Figure 2a. Pythagorean trees and fractal tilings can also be considered to be developing fractal constructs, where the development as generations are added is radial in the plane (Figure 2b). The development of fractal curves vertically but also radially results in a surface for which the topmost curves lie in roughly spherical or dome-shaped envelopes (Figure 2c). In addition to natural trees, an analog in nature is brain coral (Figure 3). Surfaces of this sort can be more interesting visually and more compelling as sculptural forms than flat surfaces. Some different approaches to achieving this are described in the following sections.

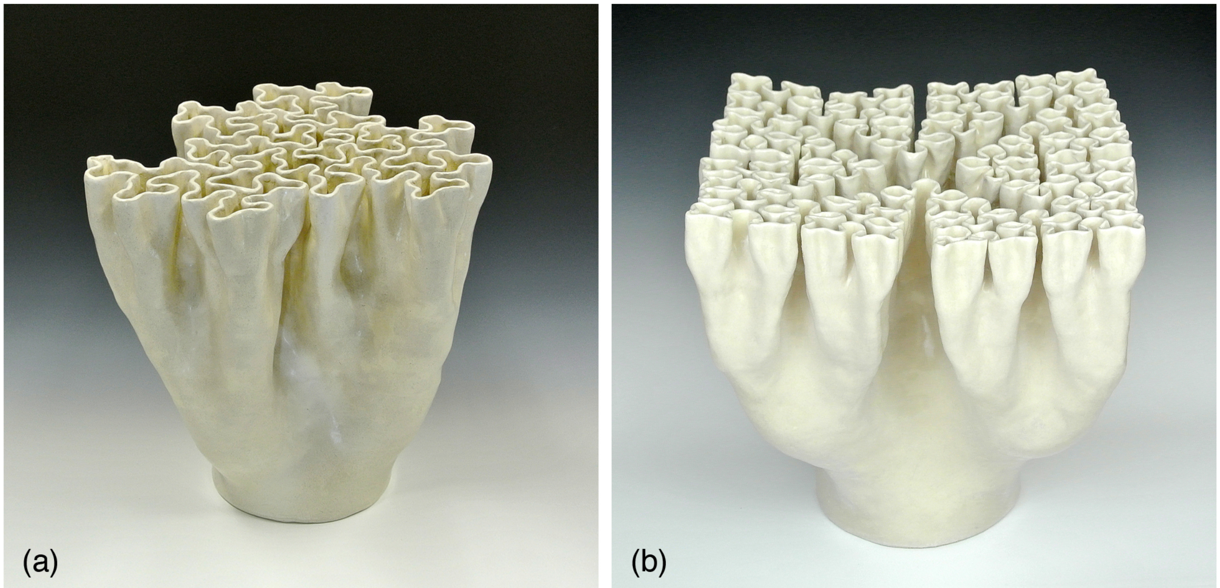

Figure 1: *Two ceramic sculptures by Robert Fathauer based on the development of fractal curves in a direction orthogonal to the plane of the curve. a) Three-fold Development (2013), and b) Four-fold Development (2014).*

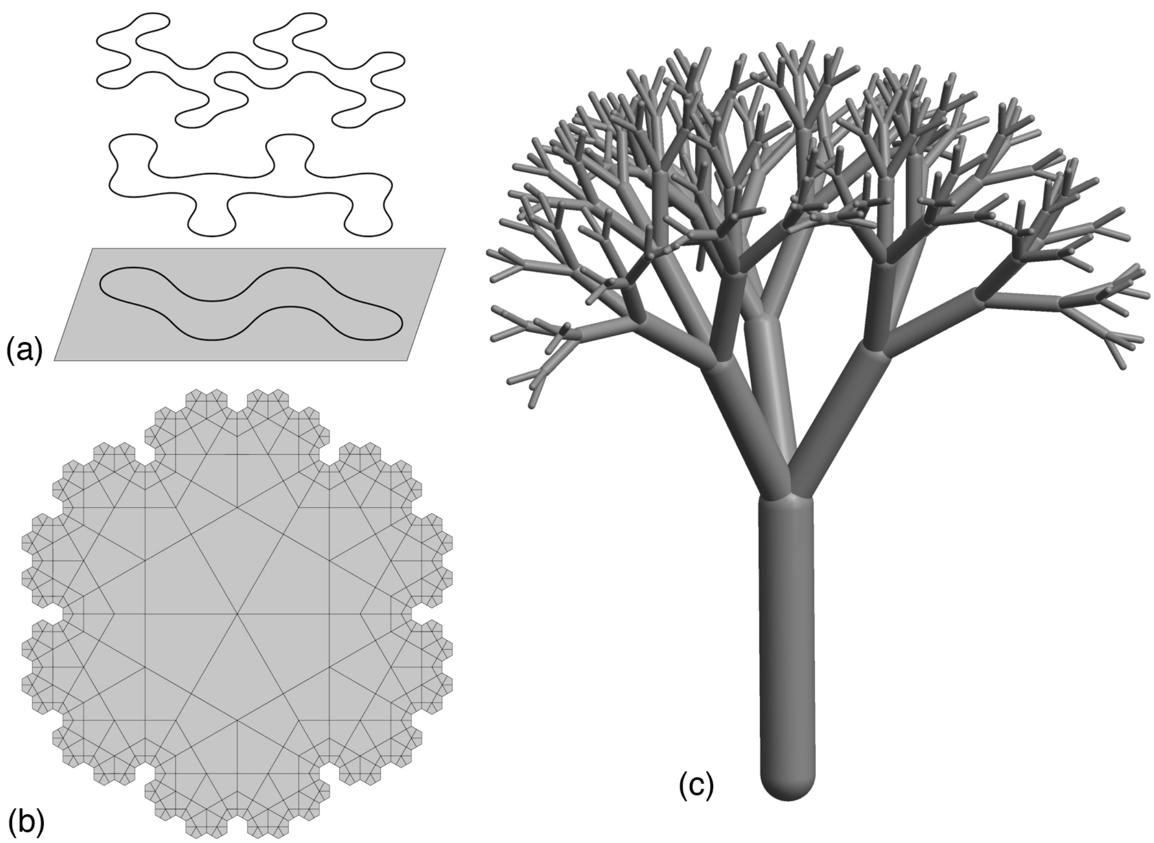

Figure 2: *Spatial development of fractals a) linearly in a direction orthogonal to the plane of the curve, b) radially in the plane, and c) radially in three dimensions.*

Figure 3: *Brain coral is a natural structure with a convoluted curve bounded by a roughly spherical envelope.*

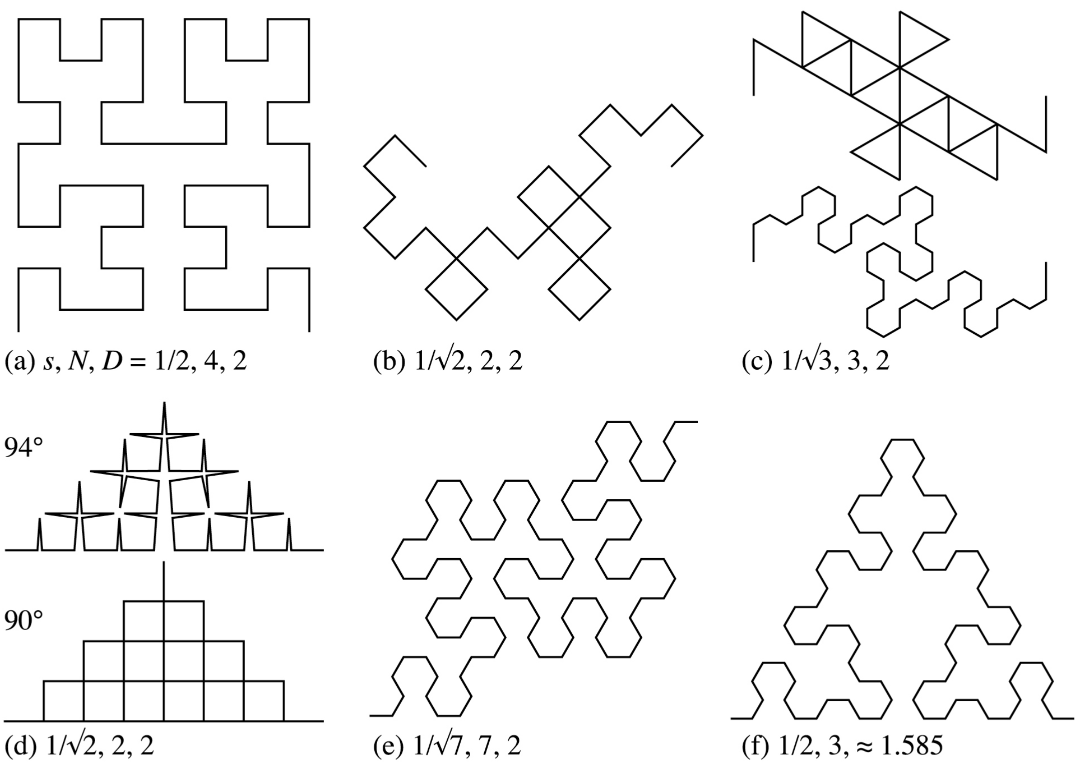

Figure 4: *Fractal curves discussed in this paper: a) Hilbert curve, b) Dragon curve, c) Terdragon curve (canonical triangle-based version on top, hexagon-based version on bottom), d) Cesàro curve (the top version is shown to illustrate the path of the canonical 90° version), e) 7-dragon curve, and f) Sierpinski Arrowhead curve.*

Several fractal curves are employed in creating these sculptural forms. Figure 4 shows these, along with the linear scaling factor s between successive generations, the increase in the number of features N between successive generations, and the fractal dimension D of the curve. The self-similarity dimension is used for D, given by $\log N / \log s$. Note that five of the six curves shown are plane filling, with $D = 2$. The Sierpinski Arrowhead curve, which is not plane filling, has a smaller fractal dimension.

Sculptures Based on Models with Polyhedral Envelopes

The different generations of at least some fractal curves can be used to cover a polyhedron with a single line. Two examples are shown in Figure 5, where the first four generations of a Hilbert curve are applied to the faces of a cube, and the first three generations of the hexagon-based 7-dragon curve are applied to the faces of an octahedron. The second of these was modified for use in a ceramic sculpture based on half the covering of the octahedron, with the first generation forming a two-lobed smooth curve, as shown in Figure 6a. The sculpture, entitled "Radial Development" (2014), is shown in Figures 6b-d. The smooth surfaces of the sculpture were worked out by starting with a saddle and then building outward to form the next two iterations of the curve. Keeping the curvature negative everywhere was a goal in this process, but there are obviously some areas where the curvature is positive. This resulted from the difficultly in creating a three-dimensional surface from curves printed on an octahedron.

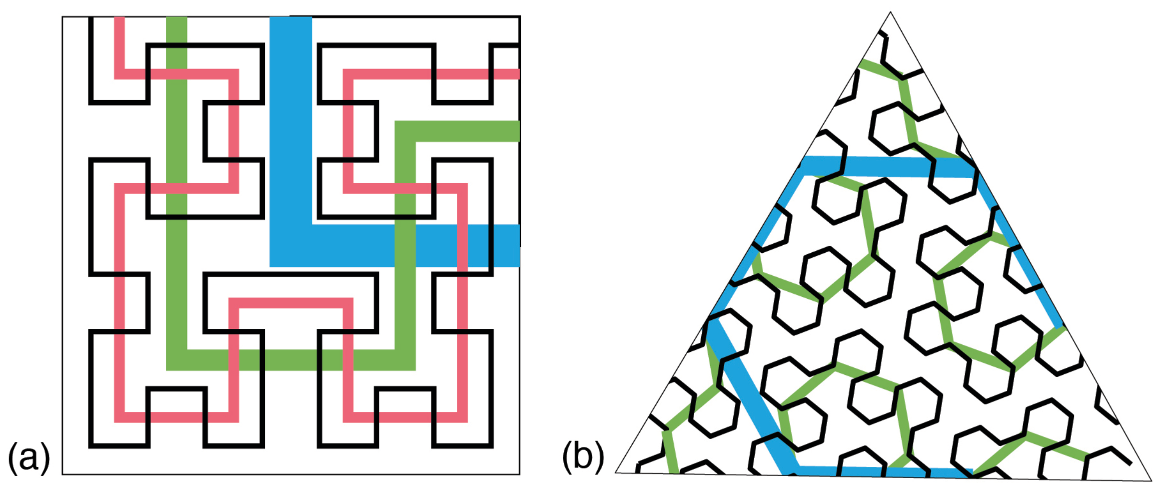

Figure 5: *a) Hilbert curve applied to a cube (a single face is shown). b) The hexagon-based 7-dragon curve applied to an octahedron (a single face is shown).*

The Hilbert-curve covered cube of Figure 5a was subsequently used as the basis for a 3D print with a spherical envelope. The 3D model was developed by David Bachman and Henry Segerman [1]. Python code was used to generate a Hilbert curve in a square. Additional code was then used to project the Hilbert curve onto 1/6th of a sphere with minimal distortion. Next, Grasshopper script was written to connect the six curves into one continuous curve at each generation. These were spatially separated radially, with higher generations having a larger overall diameter. Finally, an interpolated surface was created between the various levels. 3D prints of 1/3 of the structure and the full structure are shown in Figure 7.

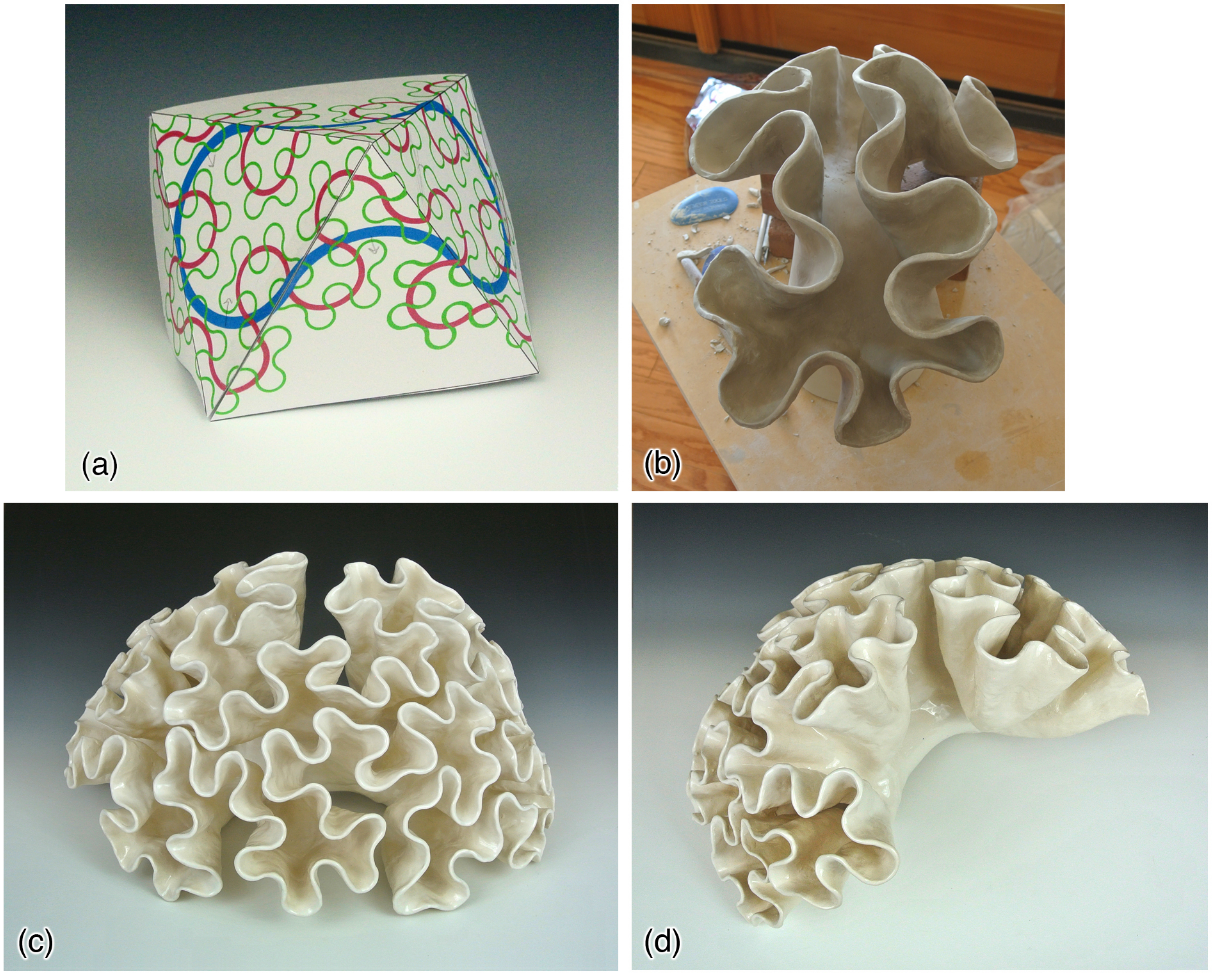

Figure 6: *a) Paper model of two-lobed smooth curve used as a basis for a ceramic sculpture entitled "Radial Development" (2014). b) The sculpture after building out to the second generation of the curve. c, d) The finished sculpture from two different angles.*

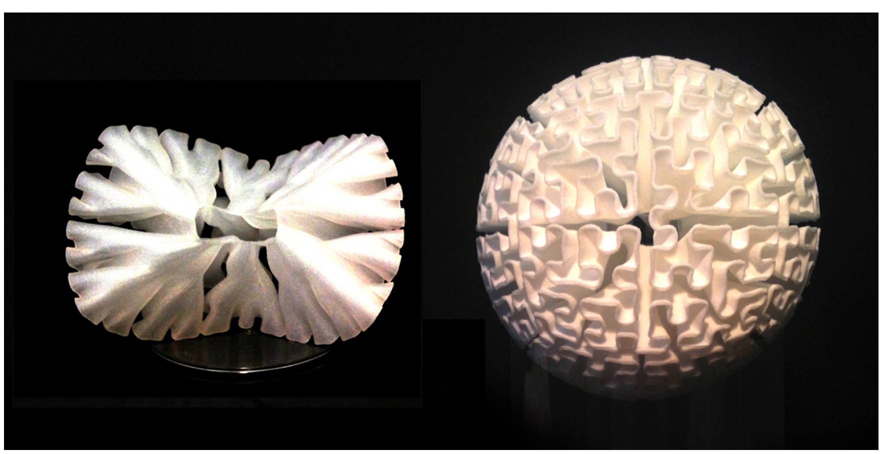

Figure 7: *Photographs of a 3D-printed radially-developing Hilbert curve, where 1/3 of the full sphere is seen at left, from the center out, and the full sphere at right.*

Structures Based on Paper Folding

As shown in [4], some developing fractal curves, such as the dragon curve, can be made in paper by folding a Pythagorean tree consisting of squares and isosceles right triangles, and then joining edges with tape. The squares provide the separation of the generations. This is readily modeled using software such as Mathematica. If the squares are replaced at each step with trapezoids that have their bases attached to the outer triangles, the structure then fans out as it grows, as illustrated in Figure 8 with a Cesàro curve. Modeling this sort of structure is much more difficult compared to the linear case, because different angles are required in different locations and generations. Using a paper construction finesses this problem because the paper flexes to adopt the necessary angles.

Multiple copies of these developing curves made with trapezoids can be used to form domed sculptural forms for which cross sections are closed curves. A domed sculptural form created by joining four developing Cesàro curves made with trapezoids is shown in Figures 8d and 8e.

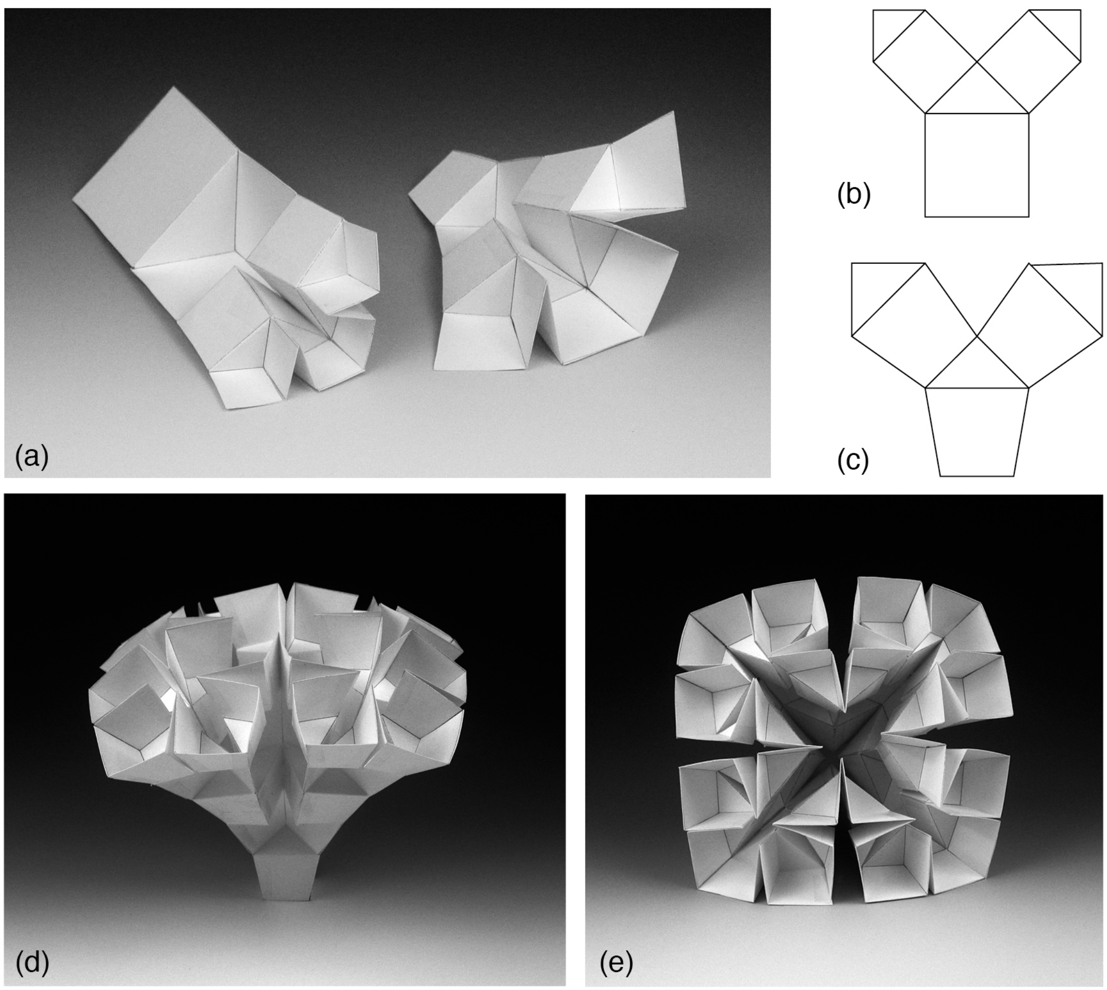

Figure 8: *a) Photograph of the first three stages of a developing Cesàro curve made from a Pythogorean tree with squares (left) and trapezoids (right). Planar building blocks for the square case (b) and trapezoid case (c). A domed structure made from four copies of the trapezoid developing Cesàro curve, seen from the side (d) and top (e).*

The above structures reveal a couple of features that make for more successful sculpture. One is the evolving structure of the surfaces being visible. This structure, which evokes some coral and tree forms, is fascinating and often beautiful. A lateral boundary of some sort in the final generation is also more engaging. The full Hilbert sphere suffers from too much uniformity, making it less interesting as sculpture. Sculpture works better when it has a markedly different appearance from different vantage points. Indeed, this is the key property making good three-dimensional forms more interesting than two-dimensional forms.

These properties were conscious goals in the following structure. A simple trifurcating tree consisting of squares and half hexagons can be folded to form a Sierpinski Arrowhead curve [4]. Attaching six of these together, alternating the side that faces inward, results in a closed form as illustrated in Figure 9a. If the squares are replaced with trapezoids, the domed three-fold sculptural form of Figure 9b is obtained. With three lobes that droop down and three that rise up to nearly meet in the middle, a graceful and varied boundary results. A natural object that shares this overall form and symmetry is an iris flower. This paper model was used to guide the constructions of a clay sculpture, where smooth surfaces were employed. The surfaces were developed in a manner that attempted to retain negative curvature everywhere. The resultant sculpture, completed in March of 2017, is entitled "Negative Curvature".

Conclusions

This paper describes an ongoing quest to use iteration of simple geometric building blocks to make models that can guide the creation of novel and compelling sculptural forms. Future work includes adapting additional forms of this sort to traditional artistic mediums such as clay. The translation of models containing sharp angles and bends to smooth and graceful curves, whether done computationally or manually, can be challenging. Over time, through working with numerous structures of this sort, the manual translation becomes easier.

References

[1] David Bachman, Robert Fathauer, and Henry Segerman. *Mathematical Art Galleries.* `http://gallery.bridgesmathart.org/exhibitions/2015-joint-mathematics-meetings/bachman-0` (as of January 30, 20t17).

[2] R. Fathauer. *Robert Fathauer – art inspired by mathematics and nature*. `http://www.fathauerart.com/` (as of January 30, 2017).

[3] R. Fathauer. Some Hyperbolic Fractal Tilings. In Gary Greenfield, George Hart, and Reza Sarhangi, editors, *Proceedings of Bridges 2014: Mathematics, Music, Art, Architecture, Culture*, pages 87-94, Phoenix, Arizona, 2014, Tesselations Publishing. Available online at `http://archive.bridgesmathart.org/2014/bridges2014-87.html`

[4] R. Fathauer. *Hyperbolic Fractal Tilings and Folded Structures.* `http://mathartfun.com/fractaldiversions/FoldingHome.html` (as of January 30, 2017).

[5] G. Irving and H. Segerman, "Developing Fractal Curves", *J. Mathematics and the Arts*, Vol. 7, pp. 103-121. 2013.

[6] Jeffrey Ventrella, *Brainfilling Curves – A Fractal Bestiary*. `https://archive.org/details/BrainfillingCurves-AFractalBestiary` (as of January 30, 2017).

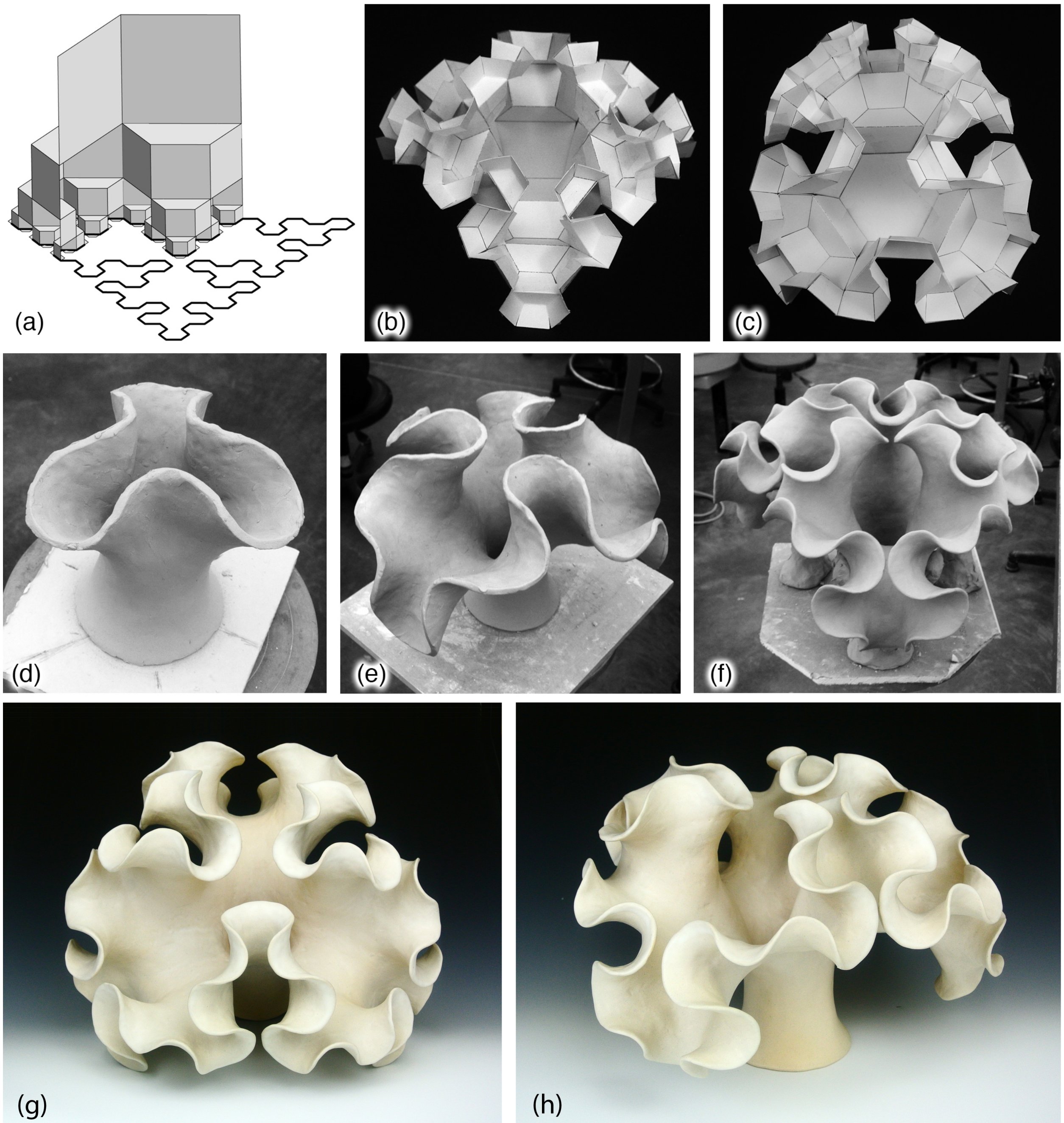

Figure 9: *a) Computer graphic of two developing Sierpinski Arrowhead curves whose top edges define two sides of a hexagon. b, c) Sculptural form created with paper by using trapezoids rather than squares in a cardstock trifucating tree, with six developing Sierpinski Arrowhead curves combined in a ring. This was carried through two generations for 2/3 of the structure and three generations for the remaining third. In b, a side view shows primarily the three-generation portion, while the top view is shown in c. d-f) Stages in the construction of a clay sculpture based on the paper model. d) Base and first generation, with three lobes. e) After completion of the second generation. f) After completion of the third generation, before firing; note the view is similar to b. g) After firing and glazing, viewpoint similar to c. h) Another view of the finished sculpture.*