Hopeless Love and Other Lattice Walks

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Abstract

The *Hopeless Love* theme arose from observations about chess bishops and their walks on the chessboard. In chess, there are two types of bishops: one confined to the white squares and the other to the black squares. If two bishops of opposite type fall in love, then they can walk around each other, coming very close together, but they can never meet. Their love is hopeless. In this article, we explore the generalization of bishop walks to the 3D chessboard and define when their paths are considered a hopeless love. This has led to several mathematical sculptures. A fictitious 3D chess piece that walks in directions of the main diagonals of the cube we call a *rhomber*. Again, hopeless love is in the picture, and has resulted in a mathematical sculpture.

1 Introduction

Figure 1 shows two wooden mathematical sculptures named *Hopeless Love I* and *Hopeless Love II*. In this article, we explain and explore the *Hopeless Love* theme. In Section 2 we tell the story of two bishops that fell in love on the 3D chessboard, and how that relates to the face-centered cubic lattice. We define the criteria for hopeless love in Section 3. Section 4 introduces a new 3D chess piece that corresponds to the body-centered cubic lattice, where hopeless love is again a possibility. Finally, Section 5 concludes the article, where we also mention some open problems.

Figure 1 : *Hopeless Love I and Hopeless Love II (wood: wenge and obeche; 35 cm high)*

2 Bishops and the Face-Centered Cubic (FCC) Lattice

The squares of a chessboard have alternating 'colors', typically called white and black, but in reality more often a light and a darker color. In the game of chess, the bishop is a chess piece that moves diagonally across the chessboard. Hence, it is confined to a single color of squares. We call that color the bishop's *type*. Initially, both chess players have one bishop of each type. Bishops of opposite types can never attack or take each other.

Mathematically, this can be described as follows. Each square on the (infinite) chessboard is identified by a pair of integer (x, y)-coordinates. The color of a square is determined by the *parity*[1] of the sum of its two coordinates. Adjacent squares (sharing a side, not just a corner) have opposite colors, and their coordinates differ by 1 on exactly one axis. Hence, they have opposite parity. The squares with an even parity we call *white*, and those with an odd parity *black*. In a bishop move, both coordinates change by the same amount (though possibly in different directions: up or down). Hence, the parity of the bishop's location does not change when it moves. That is, the bishop stays on the same color of square. Bishops of opposite type can get to neighboring squares and walk around each other, but they can never meet. If they fall in love, then this is doomed to be a hopeless love.

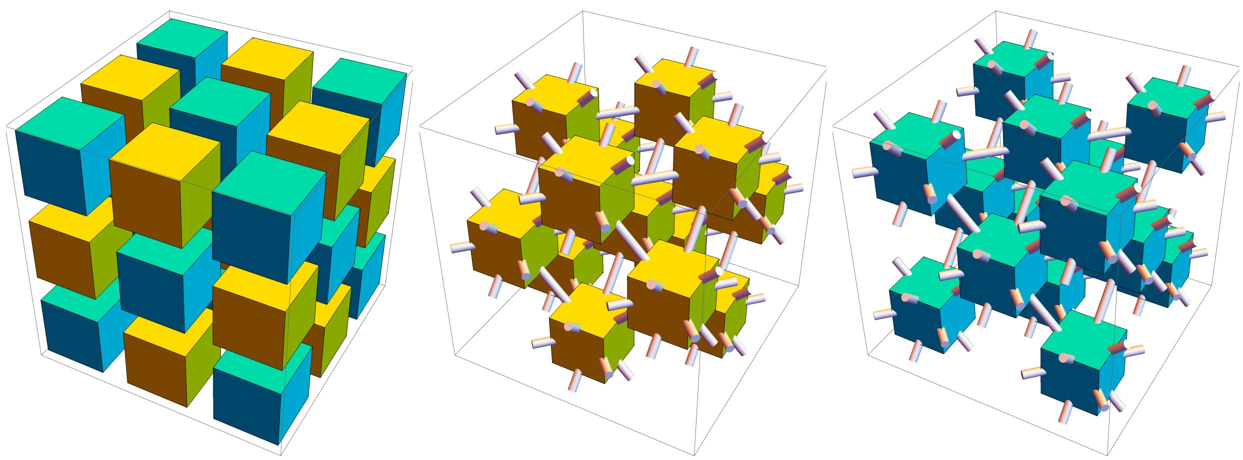


Figure 2: *3D chessboard (left) and the face-centered cubic lattice with edges (middle, right)*

Let's generalize this to space. The 3D chessboard consists of unit cubes with integer (x, y, z)-coordinate triples (see Fig. 2, left). Cubes are colored white (light) or black (dark) depending on the parity of the sum of their three coordinates. Adjacent cubes differ by 1 on exactly one coordinate axis and, hence, have opposite color. In 3D chess, the (3D) bishop moves diagonally, that is, it moves such that *exactly two* of its coordinates change by the same amount (up or down). Therefore, such a bishop is, again, confined to one cube color.

Consider a one-step bishop move: two of the three coordinates change by 1. This corresponds to an edge in the *face-centered cubic* (FCC) lattice. The 3D chessboard falls apart into two disjoint *cosets*, each with an FCC lattice structure (see Fig. 2, middle and right). As shown in [3, §3.3], a $1 : \sqrt{2}$ rectangular beam cut at $45°$ giving *square* cut faces (see Fig. 3, left) can be connected to give $90°$ *regular miter joints* and $120°$ *transverse miter joints* (see Fig. 3). It turns out that such pieces connect points in the FCC lattice, and hence correspond to bishop moves.

A bishop that walks in the 3D chessboard traces out a piecewise linear path in space. As explained in [3], closed paths are more interesting than open paths, because the latter offer no challenge when using miter joints to construct them. So, we consider closed bishop walks. Since the trapezoidal piece of Fig. 3 (a) is a nice building block, we focus on these pieces first.

[1] Whether it is even or odd.

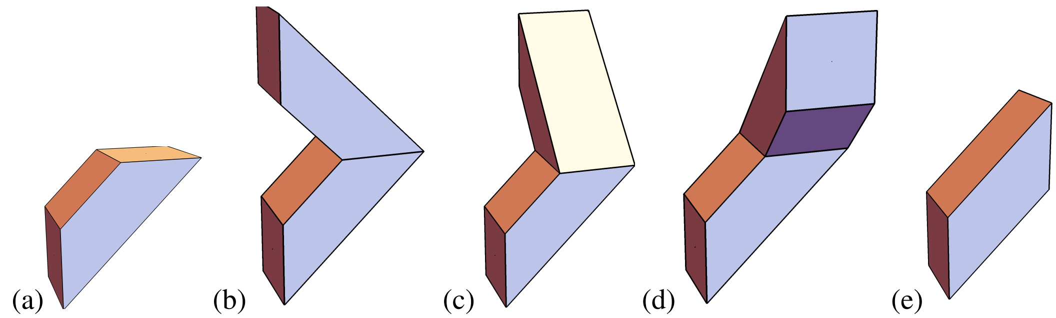
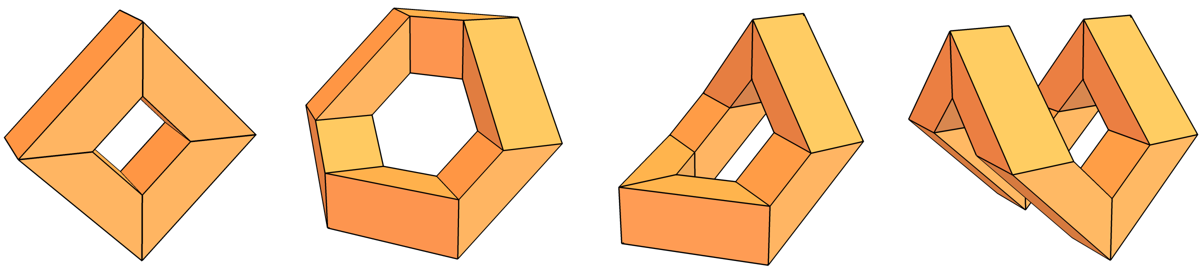


Figure 3 : $1 : \sqrt{2}$ *rectangular beam cut at* $45°$ *with square cuts: trapezoidal piece (a);* $90°$ *regular miter joint (b);* $120°$ *transverse miter joints (c, d); parallelepiped piece (e)*

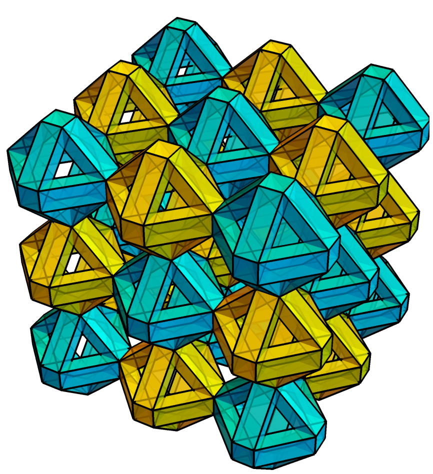


(a) (b) (c) (d)

Figure 4 : *Closed paths with trapezoidal pieces*

A natural question is: How many closed paths can you construct with n copies of this trapezoidal piece? We are interested in paths without straight connections (for aesthetic reasons, and because in that case the beam would not need to be cut at all), and without self-intersections (for practical reasons). Less than four pieces does not work, because the joint angle is at least $90°$. Fig. 4 and Fig. 6 (left) show some closed paths (also see [3, §4]). Note that the center line of Fig. 4 (b) is a *regular* hexagon, even though the hexagon may look irregular.

It can be proven as follows that a closed path always involves an even number of pieces. Fig. 5 (left) shows a superposition of all possible trapezoid paths. The trapezoids have been colored with two colors. On a path of trapezoids, these colors alternate. Hence, a closed path requires an even number of trapezoids.



Pieces: n **Closed paths:** $C(n)$

n	$C(n)$	n	$C(n)$
2	0	14	10
4	1	16	44
6	2	18	568
8	1	20	1391
10	0	22	6483
12	16	24	62882

Figure 5 : *How points in FCC lattice are visited by path of trapezoids (left); number of closed paths (right)*

Fig. 5 (right) shows some counts of closed paths, when straight connections and self-intersection are disallowed. The counts were obtained by an exhaustive search, that recursively constructs all closed paths with a given number of pieces, and then identifies those paths that are equivalent under an isometry (solid motion). Observe that the counts do not increase monotonically. Surprisingly, there are no closed paths with 10 pieces.

Note that not all FCC points can be reached via a path of trapezoids. Let a square cut face of the first piece be positioned at the origin $(0, 0, 0)$, aligned such that its normal vector points in the x-direction. Then the other square cut face can be the eight points $(\pm 1, 0, \pm 1)$ and $(\pm 1, \pm 1, 0)$. But the four points $(0, \pm 1, \pm 1)$ are not reachable, also not indirectly via other points. For the normal vector in point at (x, y, z) we have:

$(x \bmod 2, y \bmod 2, z \bmod 2)$	$(0, 0, 0)$	$(1, 1, 0)$	$(1, 0, 1)$	$(0, 1, 1)$
direction of cut face normal	x	y	z	not reachable


Figure 6 : *Sculptures based on* $1 : \sqrt{2}$ *rectangular beams with square cuts: 12 trapezoidal pieces, stainless steel, 3.2 m high (left); 12 trapezoids + 3 parallelepipeds, wood, 1.8 m high (right)*

Allowing *parallelepipedal* pieces, where the two cut faces of the beam are parallel (see Fig. 3 (e)), as well increases the number of paths (considerably). All points in the FCC lattice can then be visited, and there are also closed paths of odd length. Fig. 6 (right) shows a closed path in the shape of a trefoil knot consisting of 15 pieces, 3 of which are parallelepipeds. A construction kit with wooden trapezoidal and parallelepipedal pieces, connectable by magnets, is available under the name *MathMaker* [2].

Not all paths in the FCC lattice can be constructed from these trapezoidal and parallelepipedal pieces, because there are also FCC edges that meet at a 60° angle. They could be accommodated by cutting the rectangular beam at 30° and connecting by a regular miter joint. We have not (yet) explored this further.

3 Two Paths in Hopeless Love

Two closed bishop paths can be related in various ways. We call a pair of closed bishop paths a *hopeless love*, when the following three conditions hold.

1. Each cube visited by one path must be *adjacent* (sharing a face) with a cube visited by the other path.
2. The paths must be *topologically linked*, that is inseparable without cutting.
3. The paths are of *opposite* type; that is, the two paths are in different cosets.

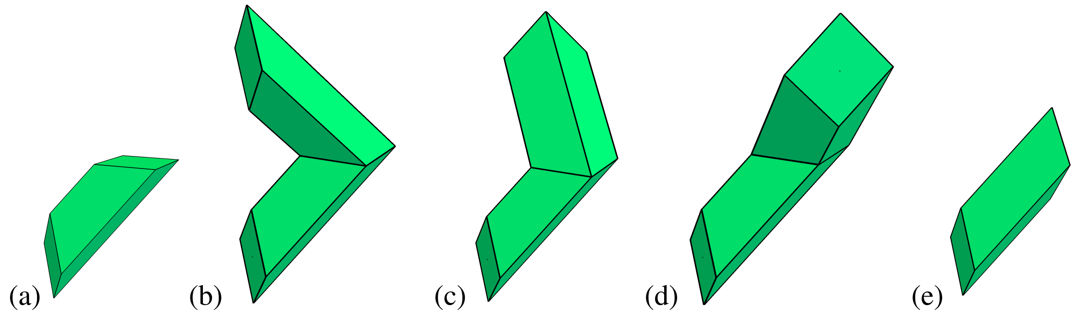


Figure 7: *$1 : \sqrt{2}$ rhombic beam with square cuts: trapezoidal piece (a); 90° regular miter joint (b); 120° transverse miter joints (c, d); parallelepiped piece (e)*

The first two conditions make it *love*: always close together and inseparable. The third condition makes the love *hopeless*. To render such hopeless loves, the $1 : \sqrt{2}$ rectangular beam does not allow elegant embracing of paths. Instead, it is better to rotate the square cut faces over 45° and work with a $1 : \sqrt{2}$ rhombic beam. Fig. 7 shows these rhombic pieces and how they connect on the square cut face.

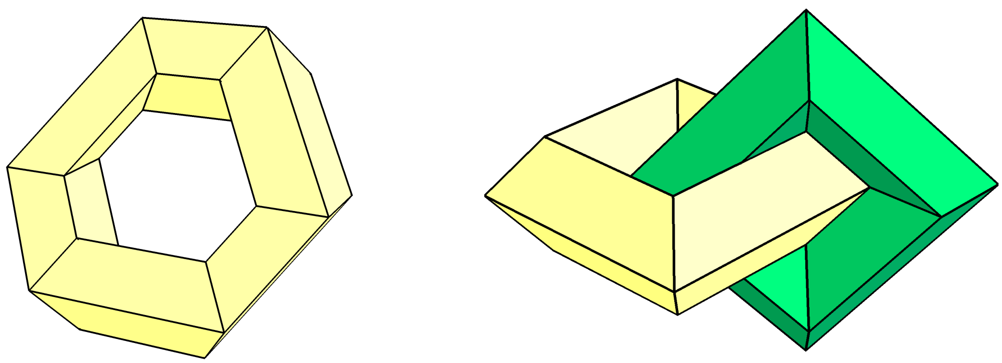


Figure 8: *$1 : \sqrt{2}$ rhombic trapezoids: hexagon (left); two squares snugly in hopeless love (right)*

Fig.8 (left) shows a hexagon made from rhombic trapezoids. Note that it now has two flat sides, because faces across the 120° joint are coplanar. Compare this to the square in Fig. 4 (a), which is also flat. However, the square with rhombic trapezoids does not have flat sides, similar to the hexagon in Fig. 4 (b). These squares snugly fit together (known as a Hopf link or $(2, 2)$-torus link), as shown in Fig. 8 (right), forming a (smallest) hopeless love.

Figure 9 provides examples of paths that are *not* in hopeless love. On the left, there is no topological linking; in fact, the paths can be moved apart without cutting them. In the middle, the two paths are inseparable, but this is metric and not topological (that is, when made from rubber, they could be separated). And on the right, there is not enough love, because the light path is not everywhere adjacent to the dark path.

Fig 1 shows two non-trivial hopeless loves. Both involve a hexagon. On the left, the black path of 12 rhombic trapezoids (cf. Fig. 6, left) winds three times around the hexagon (known as a $(2, 6)$-torus link, with linking number 3): a strong love. On the right, the white path is the rhombic version of the trefoil knot of Fig. 6 (right) with three parallelepipeds. This knot also winds three times around the hexagon: another strong love.

The hopeless loves in Fig. 1 are unbalanced, in the sense that the black and white paths have the different lengths. The hopeless love on the left in Fig. 10, which again involves a hexagon, is a $(2, 4)$-torus link. The other two hopeless loves in Fig. 10 are balanced, in the sense that the black and white paths have the same length. The one in the middle (another $(2, 6)$-torus link) has a black path consisting of 12 trapezoids, and

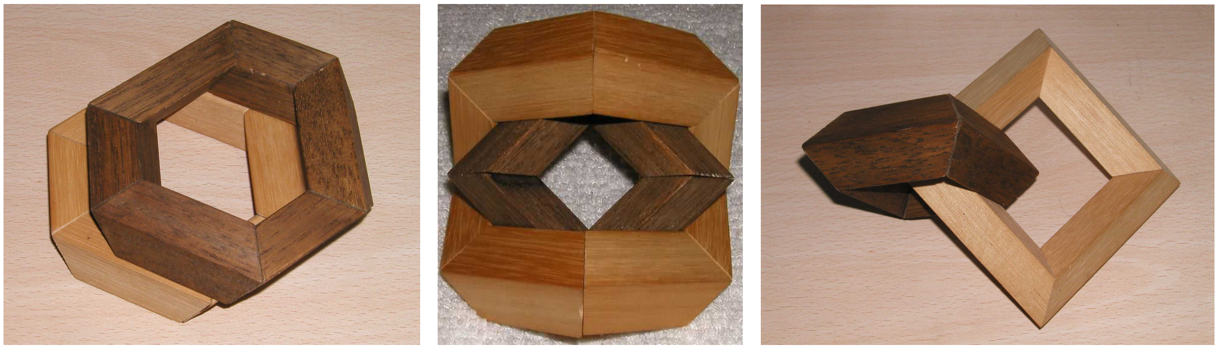


Figure 9 : *No hopeless love here: not linked (left), metrically but not topologically linked (middle), linked but not everywhere adjacent (right)*

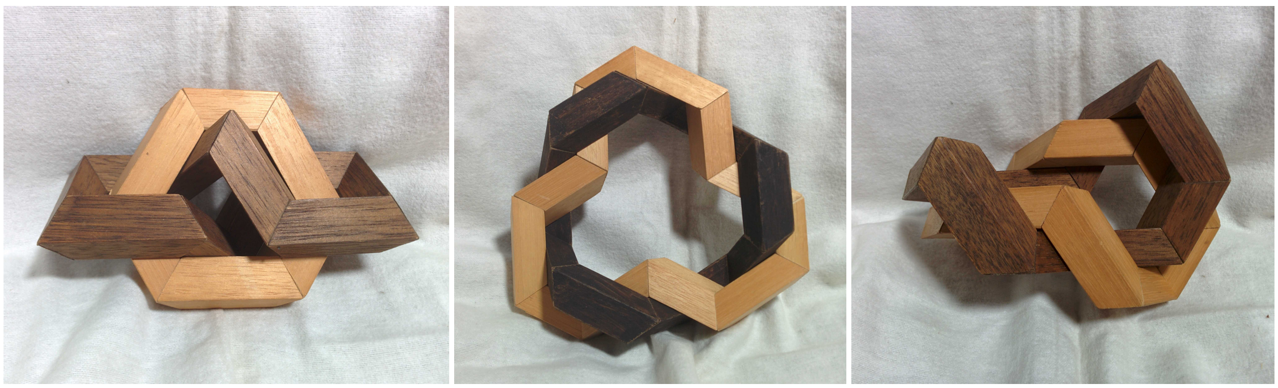


Figure 10 : *More hopeless loves: an unbalanced* $(2, 4)$*-torus link (left); black and white paths of equal length (middle, right); isomorphic black and white paths (right)*

a white path consisting of 6 trapezoids and 6 parallelepipeds (the linking number is again 3). The one on the right involves two isomorphic paths of 10 pieces each (8 trapezoids and 2 parallelepipeds; another $(2, 4)$-torus link, with linking number 2).

4 Rhombers and the Body-Centered Cubic (BCC) Lattice

When generalizing the rook to the 3D chess board, it will move in the directions of the three axes (x, y, z). During a move, only one coordinate changes. This way, the 3D rook can visit all cubes, and love between rooks is never inherently hopeless. The rook moves in the simple (or primitive) cubic (SC) lattice (see Fig. 11, left).

Another interesting lattice is the *body-centered cubic* (BCC) lattice. It has four directions, viz. the main diagonals of the cube (see Fig. 11, middle). There is no chess piece that naturally generalizes to move in this lattice. So, let's introduce a fictitious 3D chess piece that moves along the main diagonals, i.e., in the BCC lattice. We will call it a *rhomber*.[2] When a rhomber moves, its three coordinates all change by the same amount (either up or down).

It turns out that there are four types of rhombers, each confined to their own coset (see Fig. 11, right). Rhombers of a different type can walk around each other, but can never meet. So, more hopeless love lurk.

[2] In Dutch, we call it a *ruiter*, which means horse rider. That name was picked, because the Dutch word *ruit* means rhombus, and the edges in the BCC lattice form rhombuses. Moreover, rhomber sounds a bit like rider and roamer.

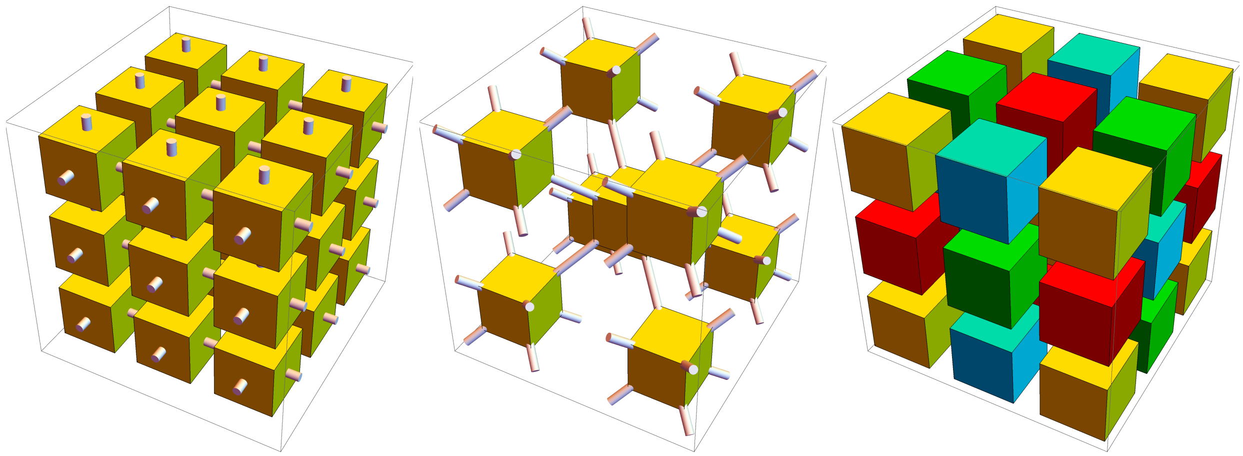


Figure 11 : *Edges in SC lattice (left); BCC edges (middle); four rhomber types (right)*

In [4], we investigated triangular beams made from $1 : \sqrt{2}$ rhombuses. These beams move in the BCC directions, and when properly scaled, they touch flush on their faces (rather than on their edges). In [4, Fig. 7 (b, d)], we encountered the doubly linked octagons. This is an example of a hopeless love in the BCC lattice. A powder-coated cor-ten steel sculpture has been based on it, called *Siamese Twins* (see Fig. 12). This involves only two of the four rhomber classes. Fig. 13 shows the two paths in the BCC lattice cosets.

Figure 12 : *Siamese Twins (two rhombers in hopeless love; powder-coated cor-ten, 100 cm high)*

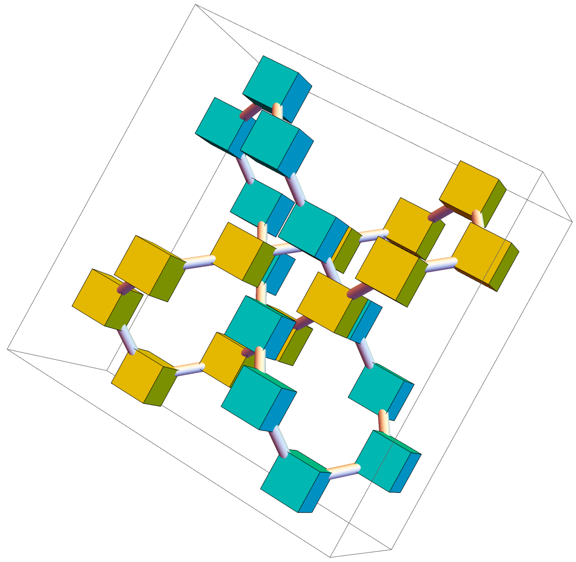


Figure 13 : *Siamese Twins BCC paths, in same orientation as Fig. 12*

5 Conclusion

We have defined and explored the hopeless love configuration, where two paths traced out in different cosets of the FCC or BCC lattice are everywhere adjacent and topologically linked. Using $1 : \sqrt{2}$ rhombic beams cut at $45°$ giving rise to square cut faces, the hopeless loves in the FCC lattice can be nicely rendered. The paths then touch flush on their faces and not just on their edges. In the BCC lattice, triangular beams accomplish the same flush touching.

Finding these hopeless loves involved an ad hoc process and some serendipity. A programmatic approach is imaginable. It could start with one closed path, next determine all adjacent cubes of one other color, and then search for closed paths through those cubes.

We are still looking for a four-fold hopeless love in the BCC lattice that involves all four rhomber types. Note that the Bamboozle [4], which arose from linking trefoil knots in the BCC lattice, does not qualify, because those knots are not every adjacent to each other.

Acknowledgments The pictures were made available by the *Foundation MathArt Koos Verhoeff* [1].

References

[1] *Foundation MathArt Koos Verhoeff* (Stichting Wiskunst Koos Verhoeff). URL: `wiskunst.dse.nl`

[2] *MathMaker*, wooden construction game, consisting of 24 rectangular trapezoids and 6 rectangular parallelepipeds, connectable by magnets, marketed by Philos. URL: `mathmaker.nl`

[3] Tom Verhoeff, Koos Verhoeff. "The Mathematics of Mitering and Its Artful Application", *Bridges Leeuwarden: Mathematics, Music, Art, Architecture, Culture*, pp. 225–234, 2008. URL: `archive.bridgesmathart.org/2008/bridges2008-225.html`

[4] Tom Verhoeff, Koos Verhoeff. "Folded Strips of Rhombuses, and a Plea for the $\sqrt{2} : 1$ Rhombus", *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture*, pp. 71–78, 2013. URL: `archive.bridgesmathart.org/2013/bridges2013-71.html`