A General Method for Building Topological Models of Polyhedra

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Abstract

We show how to build 3D topological models of arbitrary convex polyhedra by connecting same-shape pieces. We also show that many well-known tensegrity structures can be obtained as particular instances of this method. We illustrate the method with examples of models made of a variety of materials.

Introduction

We focus here on building, in a modular fashion, "topological" polyhedra models that preserve the connectivity properties between the vertices, faces, and edges, but not the spatial characteristics such as length and angles. We show that all convex polyhedra can be modeled using a single type of construction element (representing a polyhedron edge) and a single type of connection between these elements.[1]

In the remainder of the paper we introduce the construction elements, describe the models that can be built with them, and then show how these elements can be replaced with simpler, linear shapes that generate both woven-style models and tensegrity structures. In the last section we show how to modify the construction elements to bring the built objects closer to the modeled polyhedra.

George Hart pioneered modular constructions using multiple copies of a single shape (see for example [2]); many of his amazing creations fit into the paradigm described here. The sculptures described in [4] can also be derived from our method. Some of the ideas presented here were first introduced in [1].

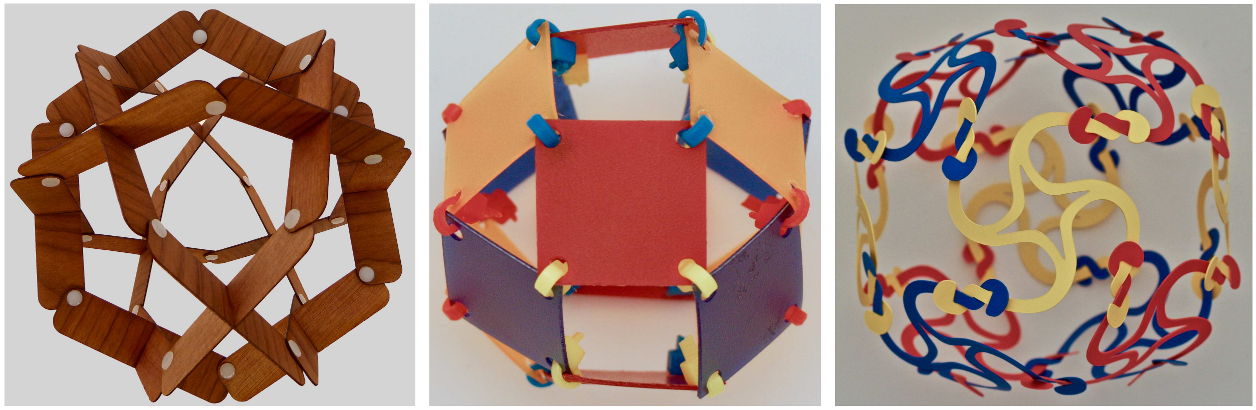
A Construction Element with Four Connection Points

We restrict the presentation to convex polyhedra; to avoid repetition, "polyhedron" will mean from now on "convex polyhedron". By "modeling a polyhedron" we mean, in fact, modeling its *polyhedral graph*, the undirected graph formed from the vertices and edges of the polyhedron; we do not distinguish between polyhedra that have the same graph. We prefer using, whenever possible, geometric rather than graph theory terms and we will use the names of various polyhedra to refer to the underlying graphs.

The construction element described in this section is characterized by four connection points. These points are usually arranged symmetrically around a common center; a typical embodiment is roughly cross-shaped with connection points at the ends of the four arms. Figure 1 shows some examples of construction elements; the pieces in (c) are similar to Edmund Harriss' four-arm *curvahedra*. Assembling a model is always possible when the construction elements are flexible (as in Figure 1 (c)) and is possible in many cases even when they are rigid but allow any angle connections (see Figure 1 (a), (b) and Figure 2 (a)).

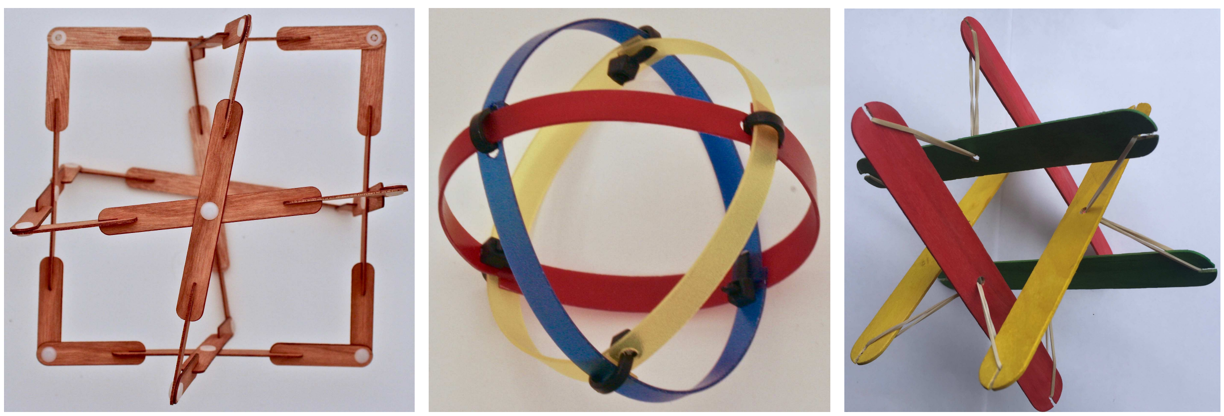
The kind of polyhedra that can be modeled depends on the interpretation given to such an element. In the "vertex" view, it represents a polyhedron vertex and the 4 half-edges adjacent to this vertex (and thus the connections between elements represent the edge midpoints). In this view we can model all elements of $\mathcal{P}_4$, the class of polyhedra where all vertices have degree 4. Conversely, every object built using such elements

[1] In general, the symmetry properties of polyhedra are also preserved by the models, but this aspect will not be discussed here.



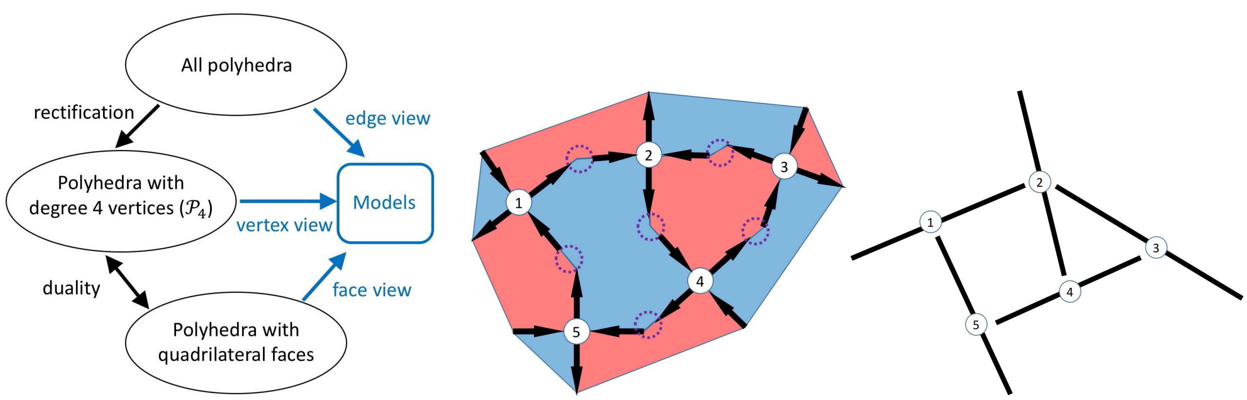
(a) Wood cross-shaped pieces (b) Rigid plastic pieces (c) Flexible plastic pieces

Figure 1 : *12 element models built with three variants of construction elements with 4 connection points*



(a) Wood pieces & hinges (b) Flexible plastic strips (c) Rigid bars (tensegrity)

Figure 2 : *Three variants of the simplest model (6 elements). Each object can be interpreted as a model of either an octahedron (vertex view), cube (face view), or tetrahedron (edge view).*



(a) (b) (c)

Figure 3 : *(a) Relations between polyhedra and models (all diagrams commute)*
(b) Vertex-view net fragment with 5 vertices and the corresponding cross-shaped elements showing the red/blue coloring and the induced edge orientation
(c) Crosses replaced by bars; vertex 2 has a complete connection (1 middle + 2 ends)

is the vertex-view model of some polyhedron in $\mathcal{P}_4$. In this view, the model in Figure 2 (a) represents an octahedron and the models in Figure 1 represent a cuboctahedron. Other examples of polyhedra in $\mathcal{P}_4$ are the rhombicuboctahedron, icosidodecahedron, antiprisms, etc.

In the "face" view, a construction element represents a polyhedron face, with the connections still at the edge midpoints. In this view we can model the polyhedra with quadrilateral faces, the duals of polyhedra in $\mathcal{P}_4$. Familiar examples include objects obtained by connecting cubes without leaving any holes. In this view, the model in Figure 2 (a) represents a cube and the models in Figure 1 represent a rhombic dodecahedron, the dual of the cuboctahedron.

Finally, in the "edge" view, a construction element represents a polyhedron edge. All polyhedra can be modeled in this way because every polyhedron edge is connected to exactly 4 other edges, where two edges are connected if they are on the same face and have a common point.[2] In this view, which is the default used in the figure captions, the model in Figure 2 (a) represents a tetrahedron. To help visualize this, we slightly distorted the cross-shaped pieces in Figure 2 (a) by reducing the angle between their arms; when the angles decrease to zero the crosses degenerate to single slats representing the edges of a tetrahedron. Similarly, the models in Figure 1 represent either a cube (in (b) imagine narrowing the red pieces to their horizontal midlines and the yellow and blue pieces to their vertical midlines with the triangular faces becoming intersection points of 3 edges) or an octahedron (imagine narrowing the pieces to the other midlines). In general, as explained below, dual polynomials generate the same model in the edge view.

The three interpretations of the model in Figure 2 (a) under these three views express the fact that the following graphs are isomorphic: (a) the graph representing the connectivity between the vertices of an octahedron (the polyhedral graph of the octahedron), (b) the graph representing the connectivity between the faces of a cube, and (c) the graph representing the connectivity between the edges of a tetrahedron. For the models in Figure 1, the isomorphic graphs are the "vertex" (polyhedral) graph of the cuboctahedron, the "face" graph of the rhombic dodecahedron and the "edge" graphs of the cube and the octahedron.

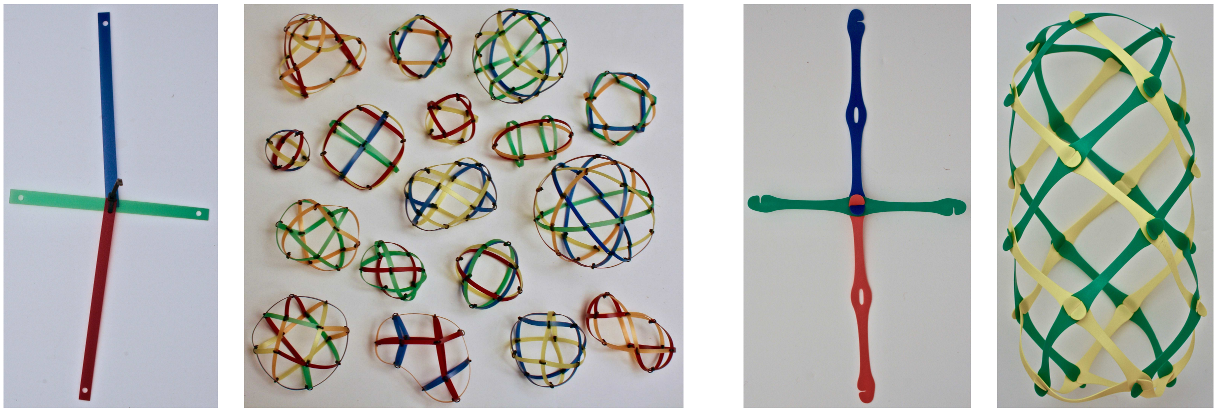
An "edge" graph can be reduced to a "vertex" graph of another polyhedron obtained via a simple construction. For a polyhedron P, its *rectification* is a polyhedron $r(P)$ whose polyhedral graph is built by choosing as vertices the midpoints of the edges of P with two midpoints joined by an edge iff the corresponding edges of P are connected as defined above. [3] Since the polyhedral graph of $r(P)$ is isomorphic to edge graph of P, $r(P) \in \mathcal{P}_4$ and the edge-view model of P is the same as the vertex-view model of $r(P)$. By rectification we obtain an octahedron from a tetrahedron, a cuboctahedron from a cube or octahedron, an icosidodecahedron from an icosahedron or dodecahedron, an antiprism from a pyramid, etc.

If P has n edges then $r(P)$ has n vertices; from Euler's formula and the fact that each vertex has degree 4, it has $2n$ edges and $n+2$ faces. If P' is the dual of P then $r(P) = r(P')$; in other words, the edge graphs of P and P' are isomorphic and thus, as mentioned before, P and P' generate the same edge-view model. This model resembles a combination of P and P' (for example, cube/octahedron in Figure 1); some ways to bias the model towards either P or P' will be discussed later in this paper.

As an interesting aside, note that for any $n \geq 8$ we can build a model using exactly n construction elements. This follows from the fact that for every $n \geq 8$ there is a polyhedron with n edges. Indeed, for $n = 8$ we have the square pyramid and assuming P has n edges we can obtain a polyhedron with $n+1$ edges as follows: if P has a face that is not a triangle we can add a new edge as a diagonal of this face. If all the faces of P are triangles, we can apply the same procedure to P', the dual of P (both P and P' have only triangular faces iff P is a tetrahedron and thus $n = 6$).

[2] This fact is used in computer graphics in the *winged edge* and *quad-edge* data structures.

[3] A graph theory theorem states that the graph constructed by rectification (or, in graph theory language, the *medial* graph) is always the polyhedral graph of some polyhedron. The converse is not true and, indeed, we can use the construction elements with 4 connection points to build objects that are not edge-view models of any polyhedron. The converse is true for planar graphs: a 4-regular planar graph is the medial graph of some planar graph (which might not be polyhedral).



(a) Strips connected with plastic ties

(b) Element that does not require extra connectors (two ends protrude through a middle)

Figure 4 : *Two variants of flexible linear pieces. The 24 piece model in (b) is the edge view of the self-dual polyhedron obtained by stacking two cubes and a square pyramid*

Figure 3 (a) shows the relations between the modeled polyhedra classes. To summarize, by using the construction elements described above we can model any polyhedron P in the edge view (or, same thing, the vertex view of $r(P)$). We can also model P in the vertex view if $P \in \mathcal{P}_4$ and in the face view if $P' \in \mathcal{P}_4$.

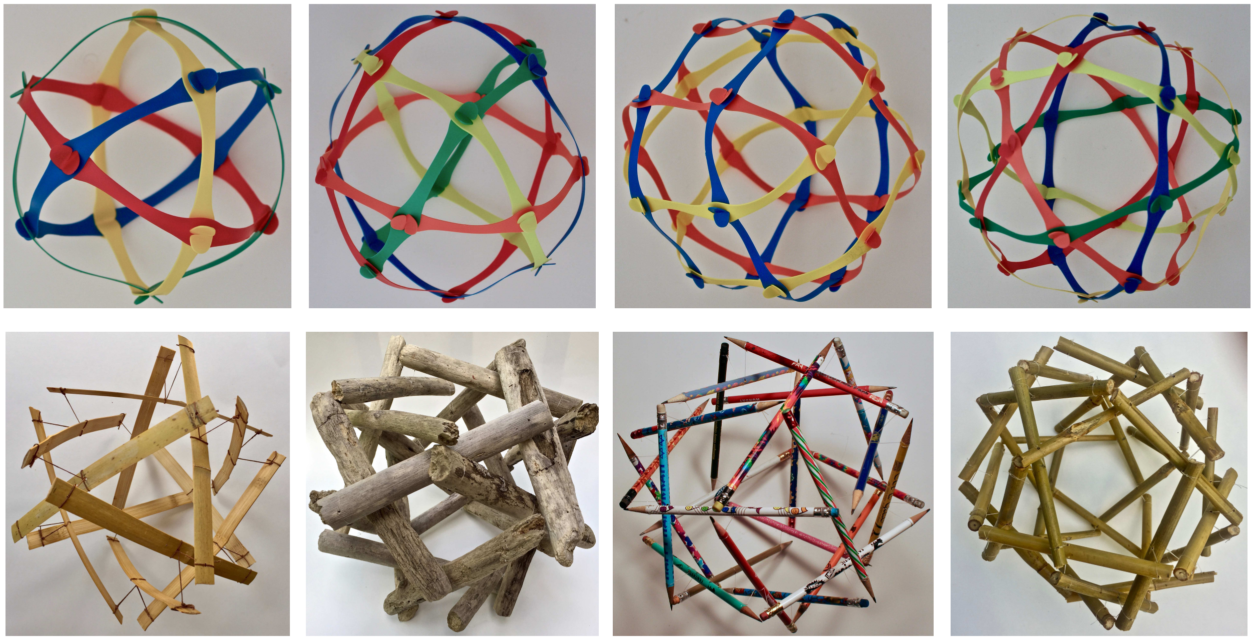
Simplifying the Construction Element

We will show now that the construction element introduced above can be replaced by a simpler, linear shape; this opens new artistic possibilities for the models.

We can abstract the construction element by a cross with connection points at the ends of the 4 (equal length) arms; as described above, in the vertex view the center of the cross represents a polyhedron vertex and a connection between two cross arms corresponds to the midpoint of a polyhedron edge. We will show that, in any model and at any vertex of this model, we can shorten two opposite arms and lengthen, by an equal amount, the other two arms. (Intuitively, we push each connection point from the midpoint of an edge towards one end of the edge; connections on two edges opposite to each other at some vertex v are pushed in the same direction, either towards, or away from, v). Indeed, let $P \in \mathcal{P}_4$. As a consequence of a well-known theorem in graph theory,[4] the faces of P can be colored with two colors, say red and blue, so that no two faces of the same color are adjacent to each other. We will orient each edge e of P such that the red face is to left of e when traveling along e in this orientation. Each vertex has now two incoming edges opposite to each other and two outgoing edges (see Figure 3 (b)).[5] We can then shorten, as described above, the cross arms corresponding to the incoming edges and lengthen the ones that correspond to the outgoing ones. At the limit, when the shortened arms have zero length, each cross becomes a single bar with two connection points at its ends and one connection point in the middle. Each bar end is connected to the middle of some other bar and each middle is connected to exactly two bar ends (see Figure 3 (c)). A polyhedron vertex v is now represented by the midpoint of some bar B while B itself corresponds to two edges adjacent to v and opposite to each other (the other two edges adjacent to v are represented by the two bars connected to the middle of B).

[4]A connected planar graph is Eulerian iff its dual graph is bipartite.

[5]When the (vertex-view) model of P is also the edge-view model of some polyhedron Q and its dual Q', i.e., $P = r(Q) = r(Q')$, the red and blue regions in Figure 3 (b) correspond to the faces of Q and Q', respectively.



(a) Cube/octahedron (12) (b) Elongated pentagonal pyramid (20) (c) Cuboctahedron/ rhombic dodecahedron (24) (d) Dodecahedron/ icosahedron (30)

Figure 5 : *Models built with flexible and rigid bars*

If the construction element is made of a flexible material, each bar middle and the two ends connected to it can be brought together to form a connection. After this transformation, the model in Figure 2 (a) could become Figure 2 (b); Figure 4 shows more models built this way. Ignoring the bar middles and following just the connections of bar ends, we can see that the bars form one or more closed circuits or loops (the loops are color-coded in Figure 2 (b) and the first row of Figure 5). When the connections are done in a uniform manner, i.e., a bar middle is always below (or always above) the two connected ends, these loops are intertwined. By using this property we can build models that keep their shape even when the bar ends are not connected to a middle but only to each other (see Figure 6).

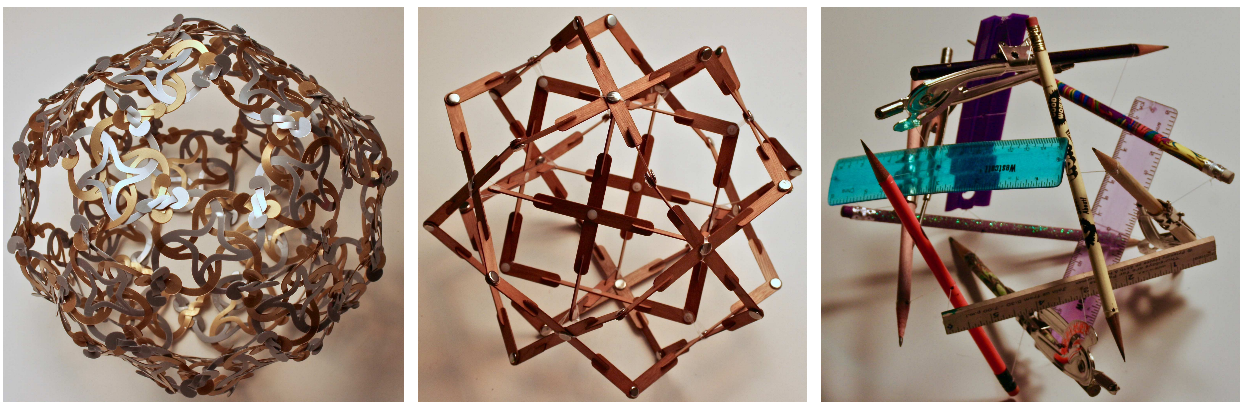
We can also use rigid or semi-rigid bars as construction elements. In this case, a middle and the two ends connected to it, although in proximity to each other, are separated and can be joined together with rubber bands (see Figure 2 (c)) or strings (the second row of Figure 5). Since the modeled polyhedron is convex, the middle will always be inside the polyhedron which causes the two strings attached to the middle to pull in opposite direction from the two strings attached to the ends of the bar. This, in turn, creates the tension that defines the shape of the model and makes it rigid; the model becomes now a tensegrity structure.[6] As mentioned before, the models built this way follow the "circuit" pattern described in [3], chs. 4,5. Many tensegrity models can be obtained in this manner; for example, by applying our method to the polyhedron defined as the rectification of a stack of similar prisms joined at their bases with, optionally, a pyramid at one end (e.g., Figure 4 (b)) we obtain the "diamond" pattern from [3], ch. 6. If two bar ends e_1, e_2 are connected to the middle m of some bar B, we obtain the "zigzag" pattern in [3], ch. 3 by moving the two connections from m to the opposite ends of B and adding a new connection (string) between e_1 and e_2.

Some extensions of the building method described here are shown in Figure 7.

[6]To create a true tensegrity model we can replace the connection (string) to the middle of a bar by two equal-length strings attached to the ends of the bar.

(a) Tetrahedron (b) Cuboctahedron/rhombic dodecahedron (c) Dodecahedron/icosahedron

Figure 6 : *Woven loops*



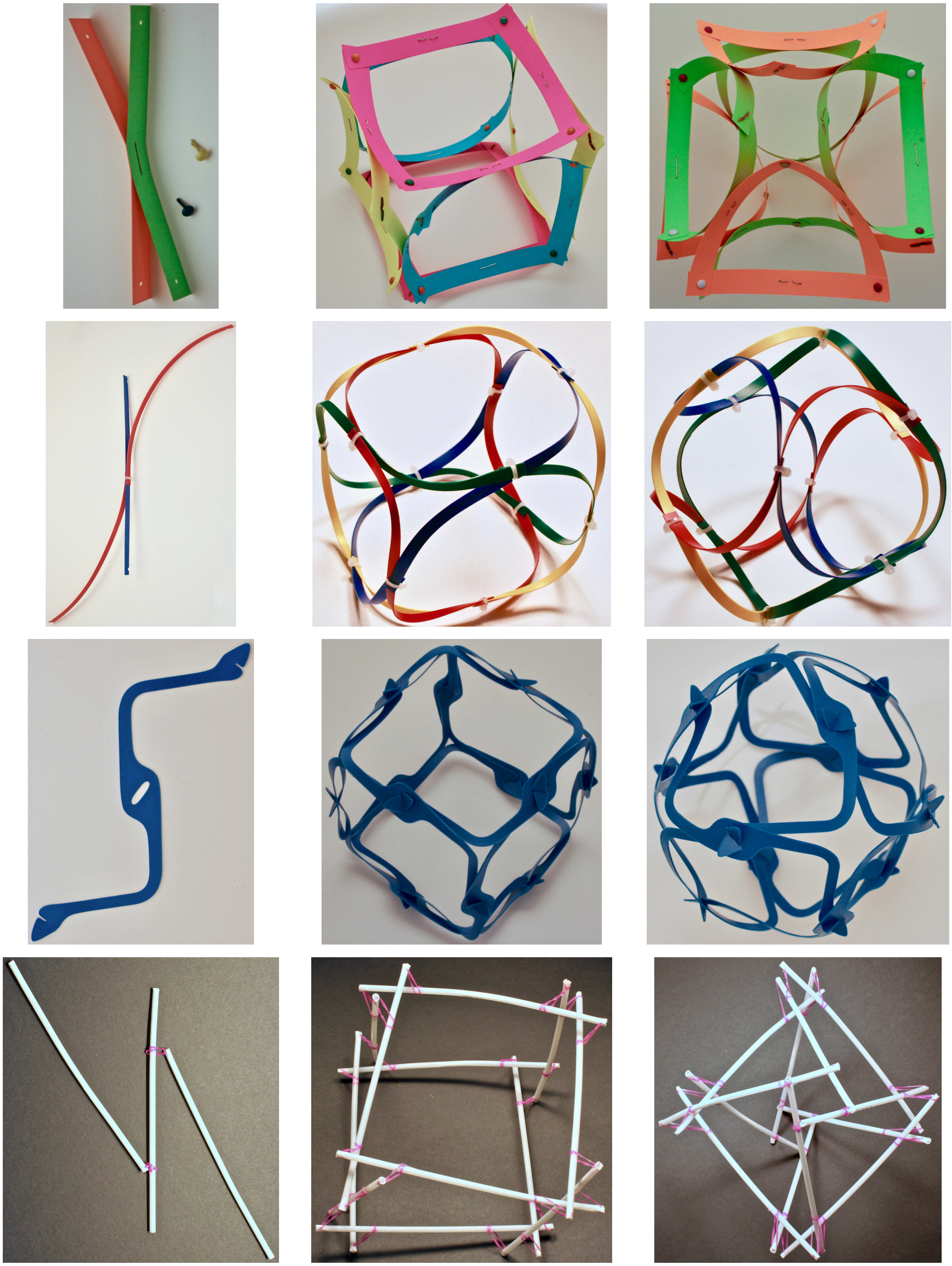
(a) Superimposed, intertwined models created by "doubling" the pieces (b) Nested models, can be extended with more layers obtained by rectification (c) *Ruler and compass construction*; different objects are used as bars

Figure 7 : *Model building variations*

Breaking the Symmetry

As discussed above, the edge-view model of a polyhedron P is the same as the edge-view model of its dual P' and the same as the vertex-view model of the rectification $r(P) = r(P')$; when built with the construction elements described above, this model will look like $r(P)$ more than P or P' (e.g., Figure 1 models recall a cuboctahedron more than a cube or an octahedron). We will show now that it is possible to build this model to resemble either P or P'; we did not have this choice before due to of the symmetry of the construction elements and the connection mechanism. If we break this symmetry by changing either the shape of the construction elements or the connection method, we can choose, when building the model, to emphasize (enlarge) the faces of P thus bringing the model closer to P (or the other way around, bringing it closer to P'). When P is self dual ($P = P'$), this choice leads to two models that are mirror images of each other.

One way to break the symmetry of the construction element with 4 connection points is to change the angle of the diagonals, making it look like an X rather than a +. Referring to Figure 3 (b), this transformation will make the red faces larger and the blue ones smaller or vice versa (depending on the orientation of the crosses). In the first row of Figure 8 we achieved this effect with a simple element made by stapling together two strips of paper. Depending on how we rotate the elements when we connect them, we can make either



(a) Construction elements (b) Cube (c) Octahedron

Figure 8 : *A cube/octahedron model in the edge view (cuboctahedron in the vertex view) can be be built to resemble either a cube or an octahedron if the the construction element is not symmetric*

the square or the triangular faces of the cuboctahedron larger. In Figure 1 (b) the same transformation can be obtained by replacing the square pieces with rectangular ones.

Moving to the linear construction elements, in the second row of Figure 8 we break the symmetry by changing the connection angle of the plastic strips (compare with Figure 4 (a)). This forces the strips to take a shape that emphasizes either the squares or the triangles.

In the third row of Figure 8, we obtain similar results by changing the shape of the construction element itself. Compare the construction element with Figure 4 (b) and compare the new models with the same model built with the symmetric version in the first row of Figure 5 (a).

Finally, when using straight, rigid bars to create a tensegrity structure, we can break the symmetry by splitting the connection point in the middle of the bar into two and moving the new connection points away from each other towards the ends of the bar (see the last row of Figure 8; corresponding symmetric models are in the second row of Figure 5 (a) and Figure 9). The distance between the connection points controls how closely the model resembles P (or P'). The tying strings become shorter as the connection points move closer to the bar ends; at the limit the object ceased to be a tensegrity structure and becomes simply a wireframe model of either P or P'.

Conclusion

We described a method for building polyhedra models by connecting, in the same manner, multiple copies of some simple shape with four connection points. By choosing different materials and connection mechanisms, this method can be used to make a large variety of geometric sculptures. The construction elements can be linear and generate either a woven-style or a tensegrity model. We thus show that any convex polyhedron has a tensegrity model and provide a unifying geometric description for many well-known tensegrity structures.

Figure 9 : *"Floating logs", by Tom Flemons*

References

[1] M. Draghicescu. Dual models: One shape to make them all. In E. Torrence, B. Torrence, C. H. Séquin, D. McKenna, K. Fenyvesi, and R. Sarhangi, editors, *Bridges Finland Conference Proceedings*, pages 635–640, Phoenix, Arizona, USA, 2016. Tessellations Publishing. Available online at `http://archive.bridgesmathart.org/2016/bridges2016-635.html`.

[2] G. W. Hart. Modular kirigami. In R. Sarhangi and J. Barrallo, editors, *Bridges Donostia: Mathematics, Music, Art, Architecture, Culture*, pages 1–8, London, 2007. Tarquin Publications. Available online at `http://archive.bridgesmathart.org/2007/bridges2007-1.html`.

[3] A. Pugh. *An Introduction to Tensegrity*. Dome series. University of California Press, 1976.

[4] D. A. Reimann. Snub polyhedral forms constructed from flexible 60-120 degree rhombic tiles. In C. S. D. M. K. F. Eve Torrence, Bruce Torrence and R. Sarhangi, editors, *Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture*, pages 443–444, Phoenix, Arizona, 2016. Tessellations Publishing. Available online at `http://archive.bridgesmathart.org/2016/bridges2016-443.html`.