Crooked Houses:
Visualizing the Polychora with Hyperbolic Patchwork

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Abstract

This paper presents kinetic models based on the 4-dimensional regular polytopes. The sequential 'flattening' is realized through the use of hyperbolic patchwork surfaces, which portray the bitruncated versions of the polychora. As pedagogical tools, these models offer a hands-on experience of 4D geometry.

Introduction

Four-dimensional space (hyperspace, 4-space) is the result of adding an extra spatial dimension perpendicular to our three dimensions of length, height and width. The research on its properties is made possible by generalizing the geometric principles acquired by studying more familiar spaces of lower dimensions. Originated in philosophy and mathematically formulated in geometry, the concept has roused interpretations in mysticism, in theoretical physics, in fiction and in visual arts. Lately the availability of digital visualization technologies has given artists a chance to study higher space with a greater fidelity to the precise geometry of the concept.

Because the fourth dimension of space cannot be directly portrayed in our physical world, the focus of many inquiries into the subject has been on the challenge of developing a visual understanding of 4-dimensional reality. Luckily, just as 3-dimensional structures can be drawn, unfolded, sliced, photographed or otherwise projected onto a 2-dimensional medium like paper or a computer screen, these graphical techniques can be generalized to produce 3D models of 4-dimensional structures described by mathematicians. The precise subject matter of these visualizations is usually the family of regular polychora – 4-dimensional counterparts of the Platonic solids. (For a detailed account of the regular polychora, see Coxeter [1] or Wikipedia [7]). In the past Bridges conferences, the topic has been treated by e.g. Saul Schleimer and Henry Segerman [3], and Carlo H. Séquin [4].

Robert A. Heinlein's 1941 science fiction short story "…And He Built a Crooked House" [2] introduces a house built in the shape of a 3-dimensional unfolding of a hypercube. As a result of an earthquake, the house folds 'up' to an actual 4D hypercube, and the inhabitants are trapped inside. Like ants walking around the surface of a cube from square to square, they navigate the eight cubical rooms of the house taking straight-line routes northeast to southwest, southeast to northwest, up and down, only to find themselves back were they started from after a round trip. Although such an experience would arguably be most unnerving, the chance to inspect the spatial interrelations of the cells of a regular polychoron would provide one with an involved insight of 4-dimensional space. This paper describes a method of facilitating such investigations with hand-held kinetic models that employ hyperbolic patchwork surfaces based on the *truncated* versions of regular polychora.

Truncation

Truncated polyhedra. The truncation[1]—as discussed here—refers to the process of cutting away every vertex of a regular solid with a plane perpendicular to a line going through the center of the solid and the vertex. As a result, the original vertices are replaced by polygons whose shape is determined by the vertex figure of the original solid. The depth of the truncation is a free parameter. Figure 1 shows truncation of a cube all the way to the dual octahedron, and the Archimedean solids met along the way. Notice how in the *cuboctahedron* the edges have shrunk to a point (rectification), and in the *truncated octahedron* the faces resulting from the truncation are already truncating each other (bitruncation). Finally in the *octahedron* also the original faces of the cube have shrunk to points (birectification).

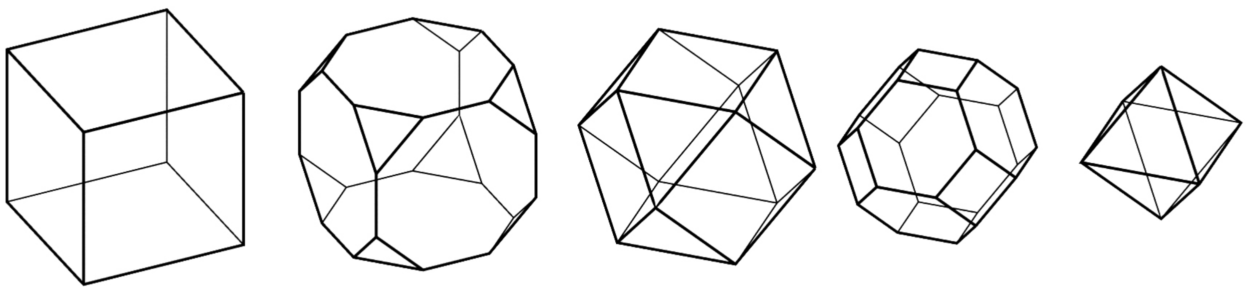

Figure 1: *Cube, truncated cube, cuboctahedron (rectified cube), truncated octahedron (bitruncated cube), octahedron (birectified cube).*

Figure 2 shows truncation of a tetrahedron all the way to the dual tetrahedron, and the Archimedean solids met along the way.

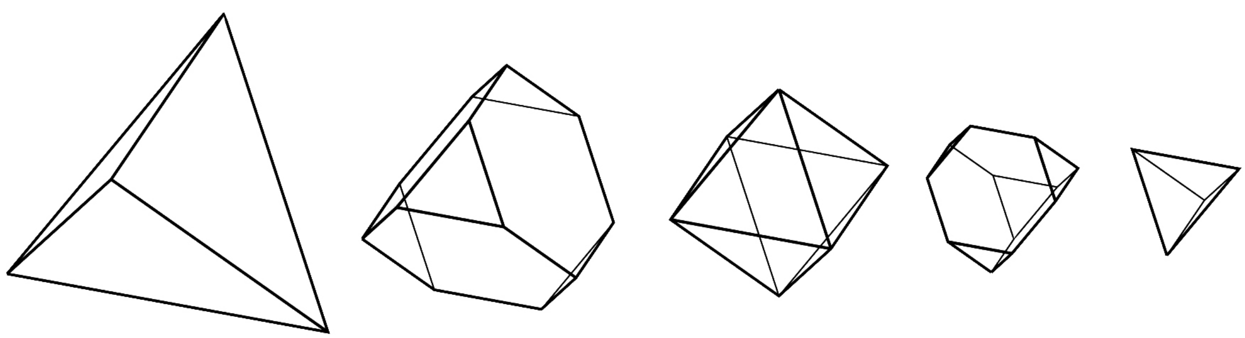

Figure 2: *Tetrahedron, truncated tetrahedron, octahedron (rectified tetrahedron), truncated tetrahedron (bitruncated tetrahedron), tetrahedron (birectified tetrahedron).*

Truncated polychora. The 4-dimensional regular polychora are truncated by cutting away every vertex of the polychoron with a *hyperplane* (3-dimensional Euclidean space) perpendicular to a line going through the center of the polychoron and the vertex. As a result, the original vertices are replaced by polyhedral cells whose shape is determined by the vertex figure of the original polychoron. The depth of the truncation is a free parameter. Figure 3 shows a sequence of truncations from the hypercube to varying depths, the new polyhedral cells highlighted in grey.

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[1] The word *truncation* is due to Kepler's naming of the Archimedean solids.

Bitruncated hypercube. For the purpose of the patchwork surface model described here, we truncate the hypercube to its bitruncated form (bottom right in Figure 3), where the cells resulting from the truncation have started to truncate each other. The bitruncated hypercube is composed of 8 truncated octahedra corresponding to the original cells of the hypercube, and 16 truncated tetrahedra corresponding to the original vertices of the hypercube.

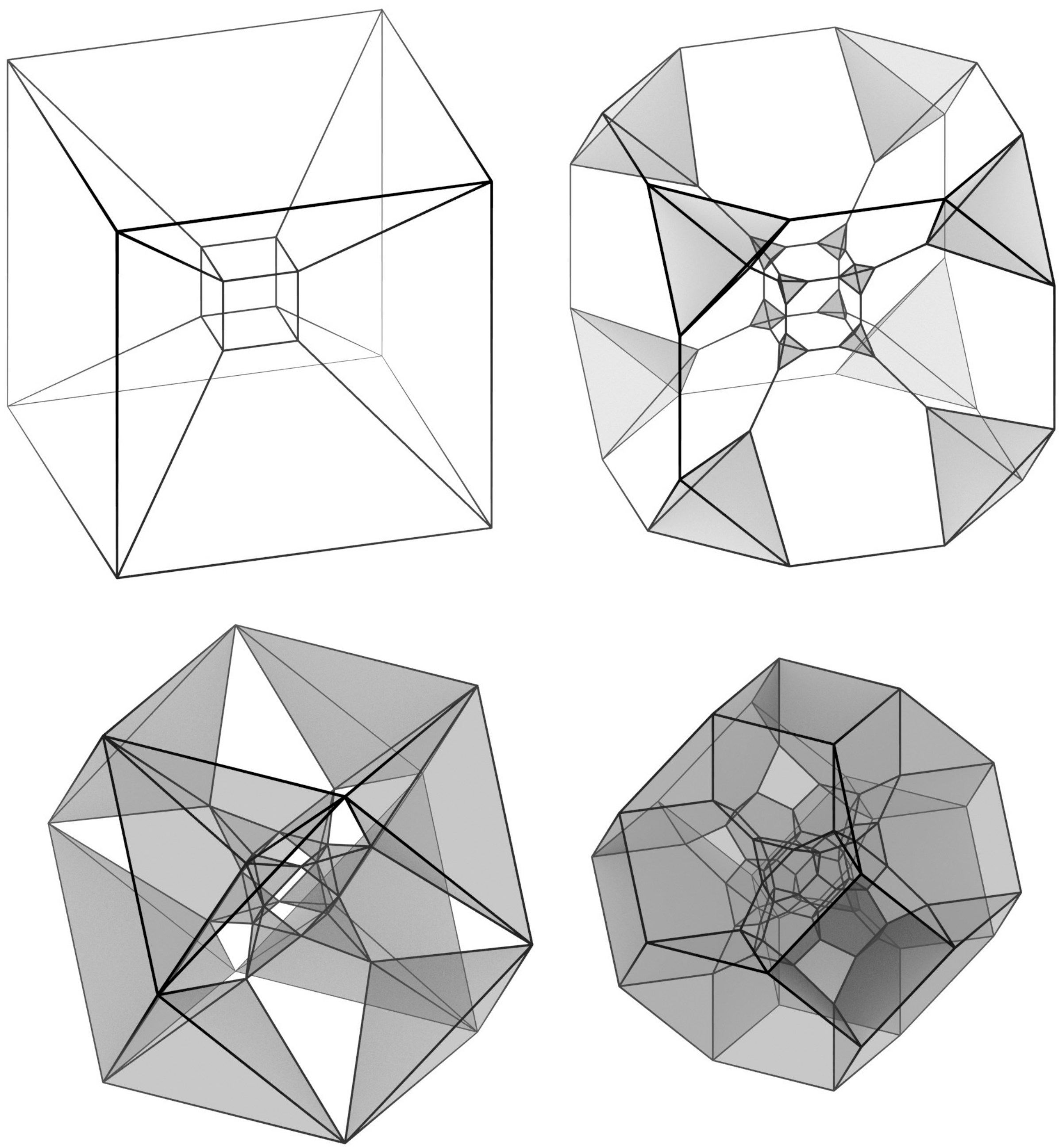


Figure 3: *Hypercube, truncated hypercube, rectified hypercube, bitruncated hypercube (perspective projections).*

We remove all the square and triangular faces of the polychoron, leaving us with a closed surface composed of 64 hexagons. As the hexagons meet 4 per vertex, the surface is *hyperbolic*.

Hyperbolic Patchwork

Colors and connections. Looking at the perspective projection of the bitruncated hypercube, it is evident that a kinetic model based on it should be able to change the size of its parts. This issue can be resolved by making the surface out of cloth, so the faces can wrinkle. The hexagons are cut from cotton cloth of eight different colors, coming in pairs of (roughly) complementary colors. The complementary color pairs chosen here are *cyan – orange*, *magenta – green*, *yellow – purple*, and *black – white*. Sixty-four hexagons are sewn together first in groups of eight to create eight truncated octahedra, each of its own color. The cells are then connected by sewing together the edges around the removed square faces of the truncated octahedra (Figure 4). The cells of complementary colors are not placed as neighbors, but as opposing cells of the structure. The finished model—the "Crooked House"—is shown in Figure 5.

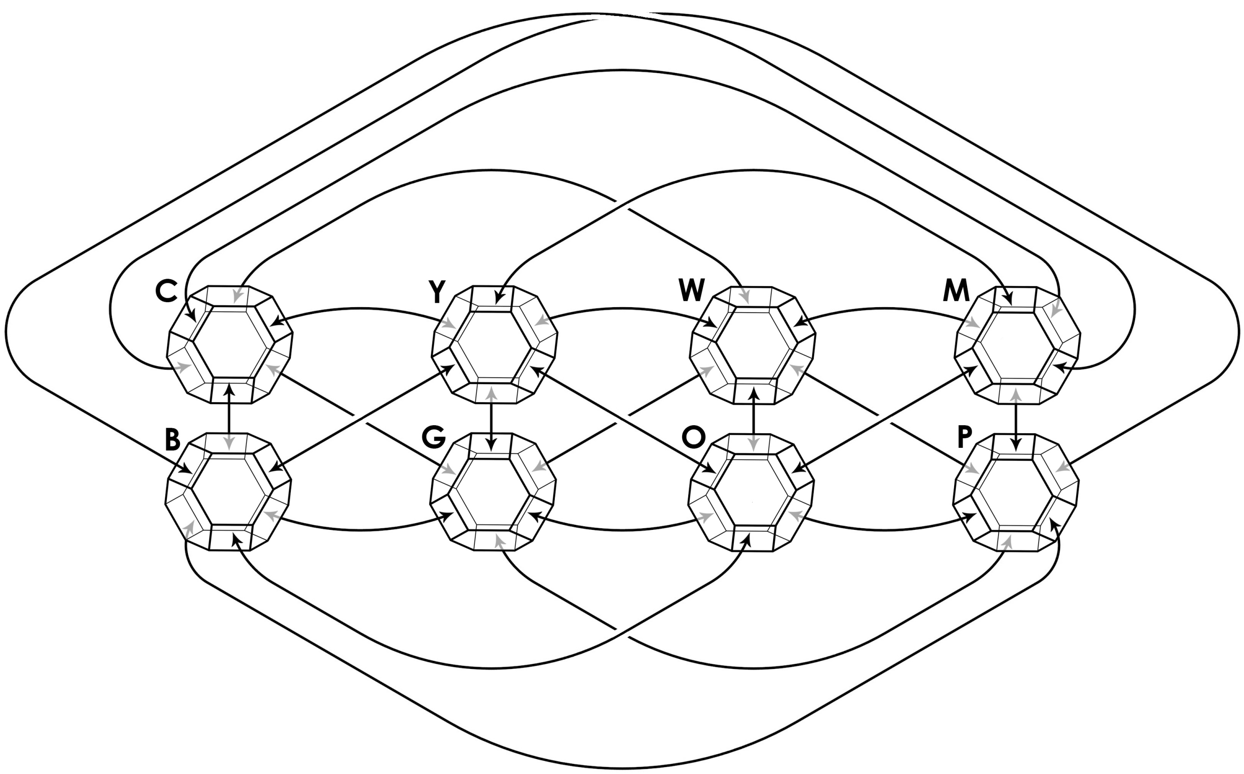


Figure 4: *Connections of the eight truncated octahedra in the bitruncated hypercube: cyan* (**C**)*, yellow* (**Y**)*, white* (**W**)*, magenta* (**M**)*, black* (**B**)*, green* (**G**)*, orange* (**O**)*, purple* (**P**).

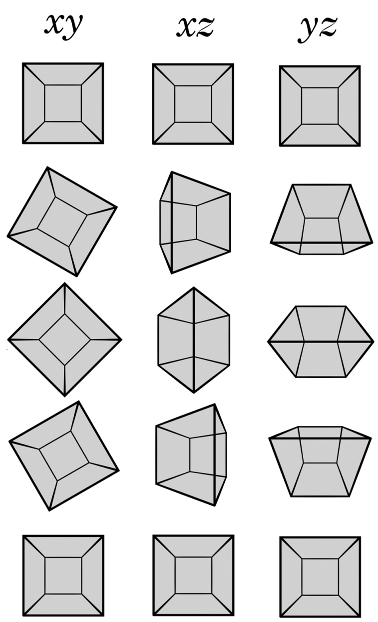
Determining the topology. The openings connecting the cells cause several handles in the patchwork surface. The exact number of handles can be verified by the *Euler characteristic* for surfaces $V - E + F$, for which we need the numbers of vertices (V), edges (E), and faces (F) of our surface. The number of the vertices must be the number of the hexagons (64) times the number of the corners in a single hexagon (6), all divided by the number of hexagons meeting at a vertex (4). Thus, the number of vertices in the patchwork equals 96. The number of the edges must be the number of the hexagons (64) times the number of the sides in a single hexagon (6), all divided by the number of hexagons meeting on an edge (2). Thus, the number of edges, or seams, in the patchwork equals 192. Together these numbers give us an Euler characteristic of $V - E + F = 96 - 192 + 64 = -32$. For closed orientable surfaces such as our patchwork[2], the relation of Euler characteristic (χ) and the *genus* (g) of the surface is given by $\chi = 2 - 2g$, so the genus of the surface is 17. This means the patchwork surface is topologically a torus with 17 handles.

Figure 5: *The "Crooked House" – A hyperbolic patchwork model of the bitruncated hypercube.*

[2] We know the patchwork is two-sided, because it has truncated octahedra on one side and truncated tetrahedra on the other. In an orientable space (like the one in which our surface is sitting), a surface is two-sided if and only if it is orientable. [6]

Planes of Rotation

Let us first examine the rotations of a cube along the three perpendicular planes *xy*, *xz*, and *yz* (Figure 6). Notice that for a 2-dimensional observer living in the surface of the picture plane of Figure 6, only the rotation along the *xy* plane would look legitimate, i.e. rigid transformation. For the 2D creature, the rotations along the *xz* and *yz* planes would seem like the figure is *turning inside out*. The square face originally on the exterior, enveloping the planar figure, is replaced with neighboring one. From our experience with 3-dimensional objects, we know that the squares appearing shrunken inside the enveloping square, are actually sticking out of the picture plane into the third spatial dimension.



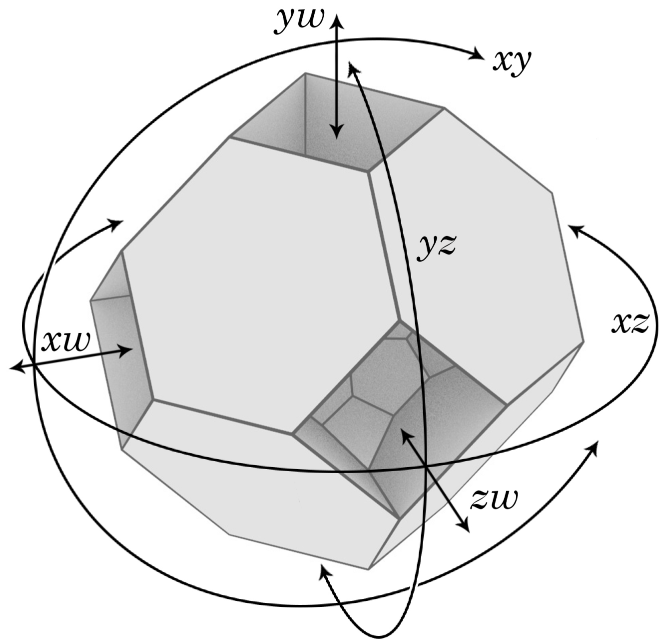


Figure 6: *Quarter rotations of a cube. along three perpendicular planes*

Figure 7: *The appearance of rotations along six perpendicular planes in the patchwork model.*

An analogous effect is witnessed in the patchwork model (Figure 7), as only the rotations along the *xy*, *xz*, and *yz* planes look like legitimate, rigid rotations. A quarter rotation along the *xw*, *yw*, or *zw* plane appers as a partial 'inside-out turning' of the surface, where a cell neighboring the enveloping outer cell is pulled out through the square opening connecting the two, and the outer cell gets pushed inside through the opening on its opposite side. The bitruncated hypercube has now changed its orientation with respect to our 3D space, and the truncated octahedron in our hands has changed its color as a consequence. The cells stuffed inside the enveloping cell should be thought of as sticking out of our space into the fourth spatial dimension.

Patchwork Visualization of Other Polychora

Besides the hypercube, we can examine the bitruncated forms of the other regular polychora to determine their suitability for hyperbolic patchwork visualization:

Hexadecachoron. The truncation of a solid will eventually yield its dual solid, so the truncation of the hypercube will result in its dual, the *hexadecachoron*, also called the 16-cell. As the bitruncated form sits between the original solid and its dual, bitruncated hypercube and bitruncated hexadecachoron are actually the same thing. This means that the patchwork model of the hexadecachoron is exactly the same surface as described above, but viewed from the other side of the surface. The volume on that side consists of 16 cells shaped like truncated tetrahedra, connected to each other via triangular openings.

The bitruncated hypercube cloth model could be equipped with an additional punctured opening allowing access to the volume on the other side of the surface. This would allow the viewer to explore the network of 16 cells of the dual hexadecahedron as well. Unfortunately this would require the patchwork model to have two fabric layers to hide the raw edges of the seams, which would make it awkward to pass all the cloth through the smaller, triangular openings.

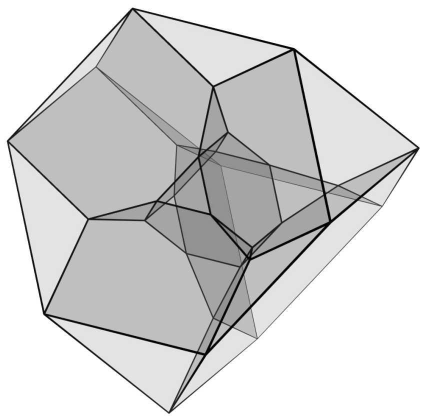


Figure 8: *Bitruncated pentachoron (perspective projection).*

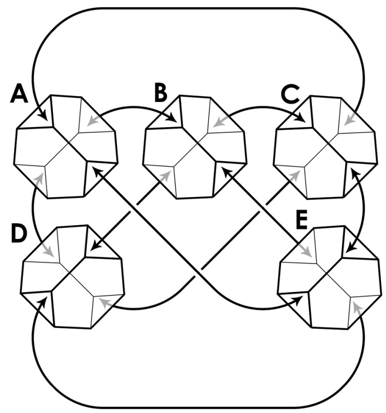


Figure 9: *Connections of the five truncated tetrahedra in the bitruncated pentachoron.*

Figure 10: *Hyperbolic patchwork model of the bitruncated pentachoron.*

Pentachoron. The bitruncation of the *pentachoron* (5-cell) results in a polychoron composed of 10 truncated tetrahedra (Figure 8). When the triangular faces are removed, the remaining 20 hexagons form a closed surface. Figure 9 shows how the cells of the bitruncated pentachoron are connected to each other via triangular openings. This surface is hyperbolic with 4 hexagons meeting at a vertex, just like the surface discussed above. As its topology is significantly simpler – a torus with six handles[3], it is clearly a suitable subject for a simple patchwork implementation (Figure 10).

Icositetrachoron. The bitruncation of the *icositetrachoron* (24-cell) yields a polychoron composed of 48 truncated cubes. When the triangular faces are removed, the remaining 144 octagons form a closed surface. This surface is hyperbolic with 4 octagons meeting at a vertex. Its topological shape is relatively complex – a torus with 73 handles. It might not be feasible as a functional patchwork model, as pushing and pulling the abundance of cloth through the triangular openings might be too laborious.

Hecatonicosachoron and hexacosichoron. The bitruncation of either the *hecatonicosachoron* (120-cell) or its dual the *hecatonicosachoron* (600-cell) yields a polychoron composed of 600 truncated tetrahedra and 120 truncated icosahedra, corresponding to the vertices of the hecatonicosachoron and the hecatonicosachoron, respectively. When the triangular and pentagonal faces are removed, the remaining 1200 hexagons form a closed surface. This surface is hyperbolic, again with 4 hexagons meeting at a

[3] The same proof for the orientability works here too, as the patchwork has 'vertex-based' truncated tetrahedra on one side of the surface and 'cell-based' truncated tetrahedra on the other.

vertex. Its topology is exceptionally complex – it is a torus with 299 handles and is not feasible as a functional patchwork model.

Conclusion

Even if the full understanding of the bitruncated hypercube patchwork model described here requires some knowledge of 4-space, I hope that it will also evoke immediate visual interest even in the lay audience. A person unacquainted with 4-dimensional geometry might describe the “Crooked House” as some kind of a color changing pouch – a 3-dimensional object interesting in its own right. In the context of 4D geometry the model presents itself as a pedagogical tool offering a chance to explore the symmetry and the interconnections of the cells of the hypercube, and to demonstrate the rotations along the six perpendicular planes concurrent on the origin in Euclidean 4-space.

Future improvements on the model would include careful consideration of a material – as seen in Figures 5 and 10, the cotton cloth gives a saggy and wrinkled appearance. Although more rigid material would articulate the cells more clearly, it might impair the movement of the surface through the openings. As for the actual geometry, making the patchwork out of hyperbolic, crocheted polygons instead of flat ones would distribute the negative curvature more evenly on the surface. Alternatively, the hexagons could be cut to have concave edges, so that the corner angles are exactly 90°. This would distribute the curvature along the edges. Using fleece instead of woven cotton cloth would let the curvature flow from the edges into the interior of each hexagon. For a hyperbolic blanket made with this technique, see [5].

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