Geometric Factors and the Well Dressed Solids of Archimedes

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Abstract

I have shown, in previous papers, that any regular polygon with n sides can be dissected into a set of isosceles triangles. These same triangles can be used to create other regular polygons with m sides provided that m is a factor of n. An enlarged version of each triangle can be created using the same isosceles triangles. In this paper I have shown how these ideas can be used to create Archimedean solids from the dissection of a single polygon. Well Dressing is a tradition in many small villages in the Pennine areas of rural England in which village wells are decorated with mosaics made from natural materials. The polygons for these solids can be in the form of an irregular tiling or a fractal. In the case of a fractal pattern I have used decorations from the Well Dressing at Hodthorpe Primary School as an inspiration for for colour choices and a source of images for decorating Archimedian solids.

Introduction to Basic Geometry of Regular Polygons

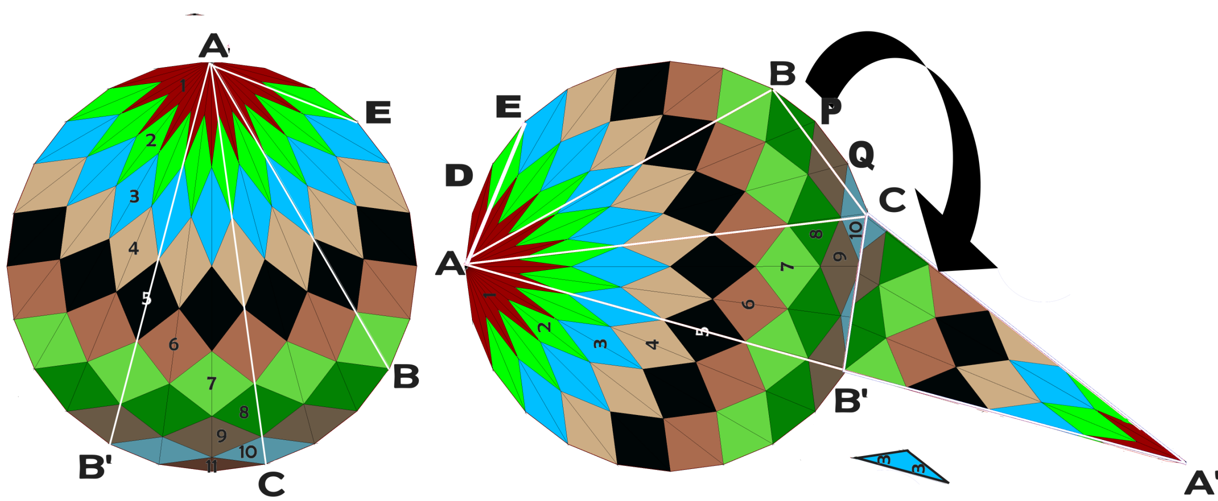

Figure 1 : *The Dissection of a Regular Polygon into its Primary Isosceles Triangles.*

Over the last few years I have developed a number of ideas concerning Precious shapes [1]. There is insufficient space in this paper to provide proof of the theorems I have used. Here, I have summarized the main points and referred to documents that give much more detail. In particular with respect to the regular polygon. The following is a list of theorems that are relevant to this paper. In each case n is the number of sides of the polygon, the side length is taken to be 1, and the angle θ, is given by $\theta = \frac{180}{n}$.

1. With reference to Figure 1, every regular polygon can be dissected into k different primary isosceles triangles, where $k = \frac{n-1}{2}$ (rounded down to the nearest integer). I have numbered these triangles 1,2 3,....k. The corresponding base angles are θ, 2θ and so on up to $k\theta$ [2].

2. I have called the line connecting two vertices of a regular polygon a diagonal. The length of any diagonal is the sum of the triangle bases that form it. In particular.

$$AE = 1 + 2\cos 2\theta \tag{1}$$

$$AC = 1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + ... + 2\cos 10\theta = 7.5958 \tag{2}$$

   AC is the largest diagonal that is less than twice the circumradius. I have called the ratio of $AC : 1$ the Precious ratio P.
3. The length of a diagonal formed by s consecutive sides is always the same. In particular $BC = B'C = AE$ where $s = 3$
4. It is always possible to create similar but larger versions of each primary triangle with a common side equal to AC, hence it will be P times larger. The angle BAC equals angle$B'AC$ =3θ. If triangle ABC is rotated about C A goes to A' and B goes to B' forming triangle ACA', an enlarged version of triangle 3. The complete set of enlarged isosceles triangles for the 24-gon are shown in Figure 2.
5. The precious ratio for the 24-gon is 7.5958

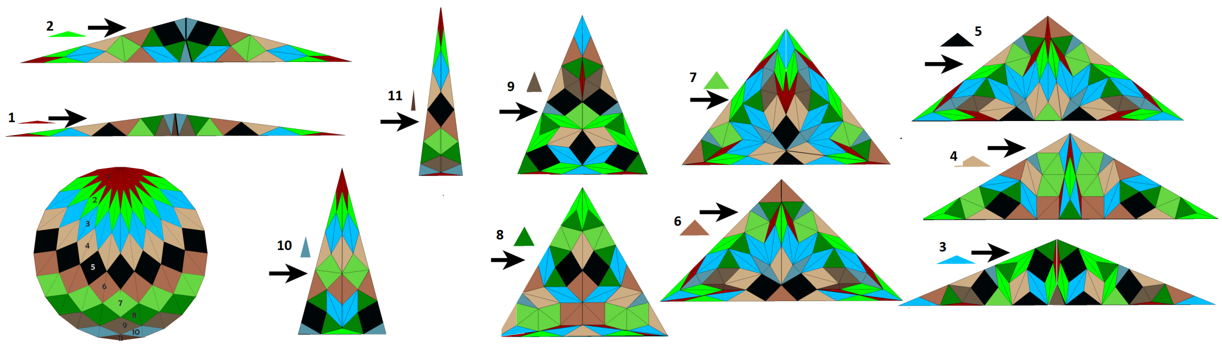

Figure 2 : *The set of enlarged primary triangles for the 24-gon. The base angles range from 1x7.5 to kx7.5 degrees. Where k=11 for the regular 24-gon.*

Geometric Factors of a Regular Polygon

1. If we consider a regular polygon with s sides then the triangles needed for its construction can be found in an n-gon when s is an integral factor of n. So a square, hexagon, octagon and 12-gon can be constructed from the dissection of the 24-gon. See Figure 4.
2. The truncated cuboctahedron, an Archimedean solid, is made from 12 squares, 8 hexagons and 6 octagons. Squares, hexagons and octagons are geometric factors of the 24-gon. Figure 3 shows half the net of the truncated cuboctahedron constructed from the primary triangles of the 24-gon. This will fold up and become the solid.
3. Figure 4 shows the dissection of the 24-gon and the construction of a square, a hexagon and an octagon using the 24-gon primary triangles. This is possible because the value of θ is an exact multiple of θ for another polygon if it is a geometric factor of that polygon [2]. For example:

$$\theta_{24} = \frac{180}{24} = 7.5 \tag{3}$$

$$\theta_8 = \frac{180}{8} = 22.5 = 3\theta_{24} \tag{4}$$

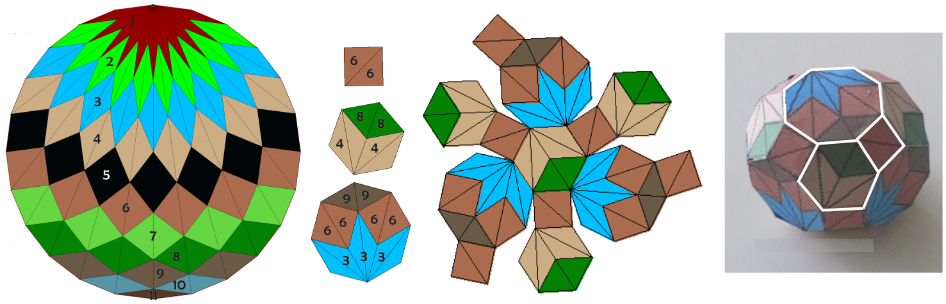

Figure 3 : *The construction of a truncated cuboctahedron from the dissection of the 24-gon*

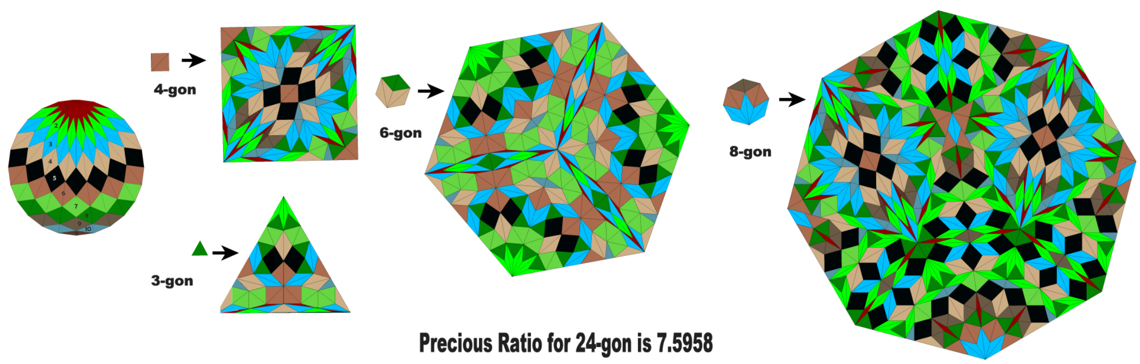

Figure 4 : *The enlargement of regular triangle, square, hexagon and octagon from the dissection of the regular 24-gon.*

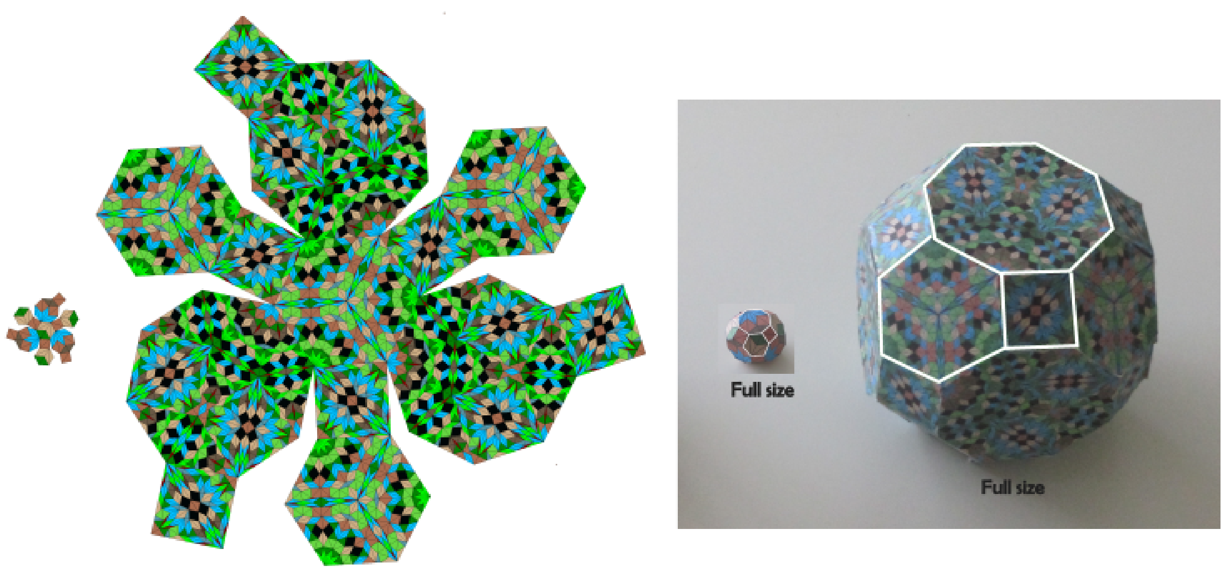

Figure 5 : *The first two models of the truncated cuboctahedron constructed from the 24-gon*

Creating an Archimedean Solid using the Geometric Factors of a Regular Polygon

The Archimedean solids are made from two or more regular polygons. We have seen that any regular n-gon, can be dissected into a number of primary isosceles triangles. In addition, we have seen that these triangles can be used to create another regular polygon, say an m-gon, providing that m is a factor of n [2]. In Figure 5 we see how a truncated cuboctahedron is constructed from the primary triangles of the regular 24-gon. The square, hexagon and octagon being geometric factors of the 24-gon. Since the primary triangles of any regular polygon have a precious relationship, similar, but larger isosceles triangles can be created from the primary triangles, see Figure 2. This implies that any shape that is made from these primary triangles can also be enlarged, see Figure 4 for the square, hexagon and octagon. Further, since these enlarged versions are made from the same primary triangles, the process can be repeated ad infinitum see Figure 4 and Figure 8.

Well Dressing Mosaics and Colour Palettes

The colour palette adopted is formed around the equilateral triangle, see Figure 6. The palette depicts the industrial heart of Hodthorpe and its surrounding natural setting. There are three colours resembling coal, dolomite and red sandstone, which have been mined or quarried in the area for many years. The outer colours green, blue and dark red represents the trees, the northern sky and the winter berries. The other colours are mixed from these basic colours. The palette can be rotated, or reflected similar to the style of Juan Gris [3]. Well dressing is a summer custom practised in rural England, in which wells, springs or other water sources are decorated with mosaic designs created from flower petals or other natural materials. The custom is most closely associated with the Peak District of Derbyshire and Staffordshire. To dress a well one or more wooden boards, perhaps 4 feet wide, and over an inch deep, are constructed. The board(s) are taken to the local river or pond and soaked for several days to ensure that they are really saturated. Then they are covered with a layer of soft, wet clay onto which the artist etches the design. The outline is the first thing to be done, followed by the 'colouring in'. The materials used vary from village to village and depend also on the time of year that the dressing takes place. For instance, a village that dresses its well in May may use blossoms and flowers whereas later in the year, seeds and berries are used as these are more plentiful. The early Christians were not happy with the custom of dressing wells - they considered it water worship and promptly put an end to it! The tradition refused to die. Tissington was the first village to re-introduce well dressing in 1349, after the village managed to escape a terrible outbreak of the Black Death that swept through England at this time. More details of well dressing pictures and methods can be seen at Glyn William's comprehensive websites [4], [5], and Rod Kikpatrick's video is availeable at [6]. These activities often include children such as scout and guide groups, schools or church groups. The examples shown in Figure 7 are extracted from Hodthorpe Primary School's well dressing.

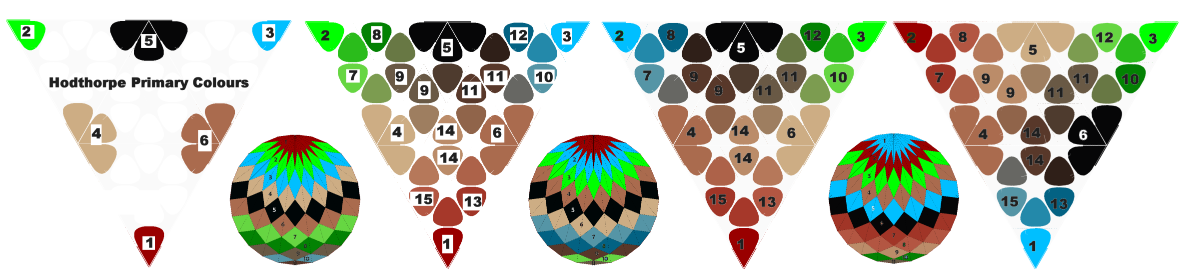

Figure 6 : *The Hodthorpe Pallette with examples of a rotation and reflection.*

Figure 7 : *Extractions from Hodthorpe Primary School's Well Dressing used to decorate fractals*

Fractal Version of Archimedean Solids

The polygonal patterns can be in the form of a Fractal. For a tiling in the plane each triangle is enlarged by the precious ratio and then decomposed into its primary isosceles triangles. For a fractal the same process is followed except that for at least one of the triangles no decomposition occurs. This leads to a fractal dimension between 1 and 2, confirming its fractal nature [2]. This is what happens for the Sierpinski seive. To illustrate this I have chosen the truncated tetrahedron, composed of four hexagons and four equilateral triangles, which are geometric factors of the regular hexagon. This has a small precious ratio so the construction of several generations of the model is simplified. Figure 8 shows the precious relationship between the primary triangles, the plane tiling of the four generations of the net, and four generations of the assembled models. An important tool for analysing such fractals is the Precious Matrix M. This shows the relationship between each Primary Triangle and the subsequent expansion and decomposition. M can be used to determine the fractal dimension. The mathematics is similar to the analysis of Route Matrices and Connectivity. Figure 9 show an undecorated fractal version. Figure 10 shows a fractal version decorated with an image from Hodthorpe Well Dressing. Further examples based upon the 24-gon, 15-gon and 12-gon can be seen in Figures 11, 12 and 13.

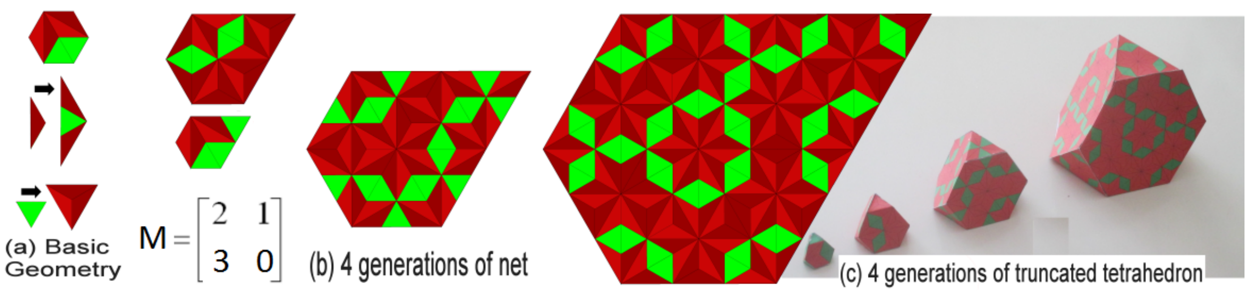

Figure 8 : *A plane tiling based on the hexagon and the truncated tetrahedron*

The Lowest Common Multiple for each Solid Shape.

I have developed the precious relationship for each of the polygons up to the 30-gon. This allows the creation of fractals and tilings for many of the Archimedean solids, see Figure 14. The development of the 60-gon is

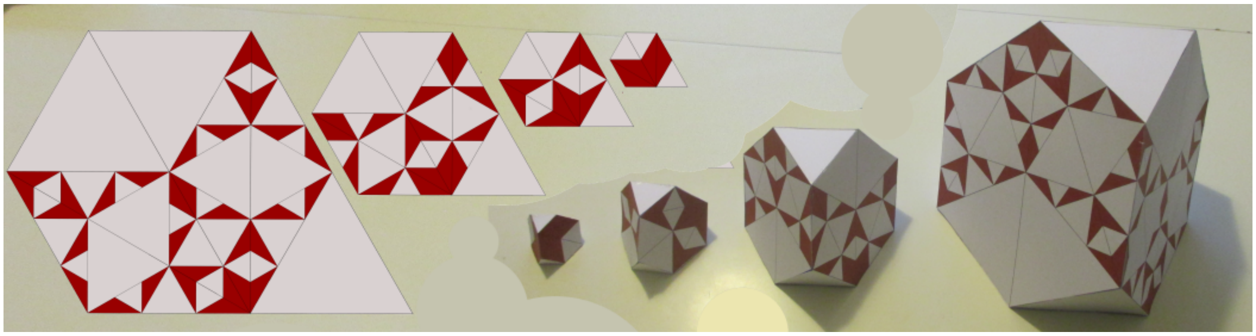

Figure 9 : *Formation of a plane fractal based on the truncated tetrahedron*

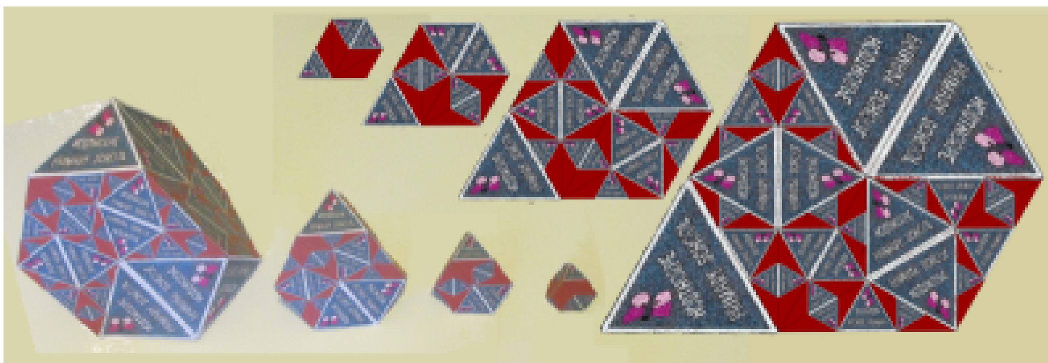

Figure 10 : *The Decoration of a fractal using an image from Hodthorpe School Well Dressing.*

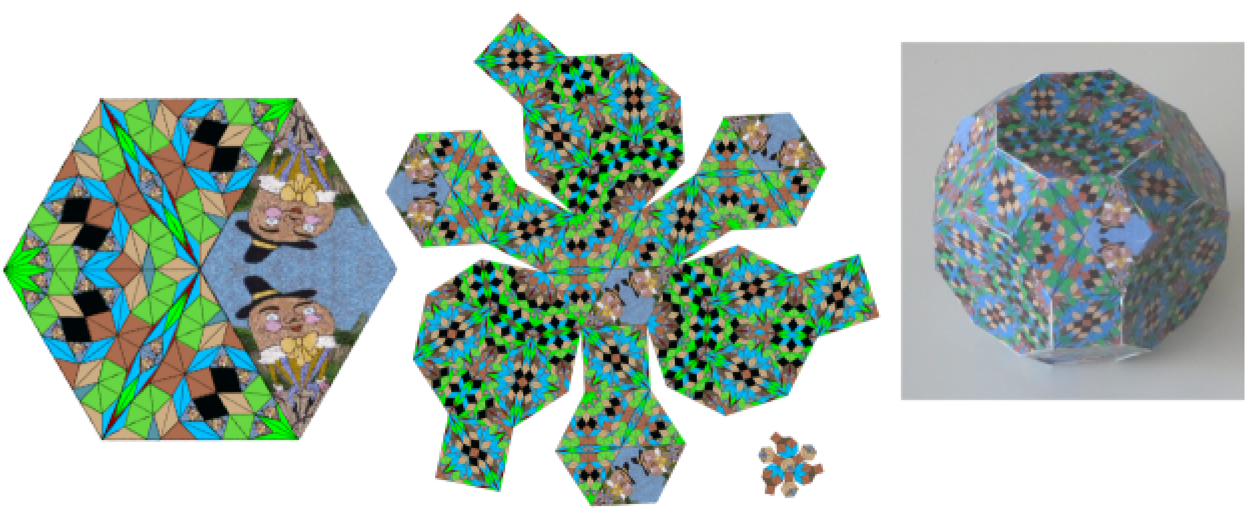

Figure 11 : *The cuboctahedron fractal decorated with Humpty Dumpty from The Hodthorpe Well Dressing, based on the 24-gon*

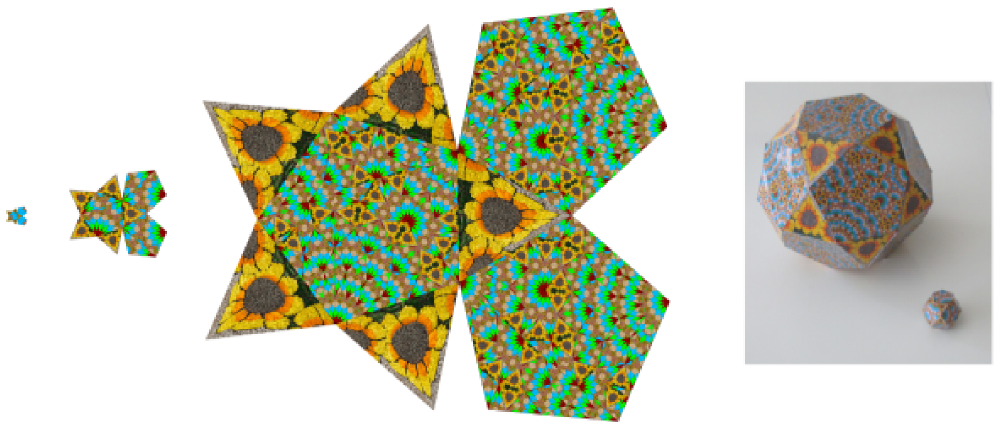

Figure 12: *The truncated dodecahedron fractal decorated with a sunflower design from the Hodthorpe Well Dressing based on the 15-gon.*

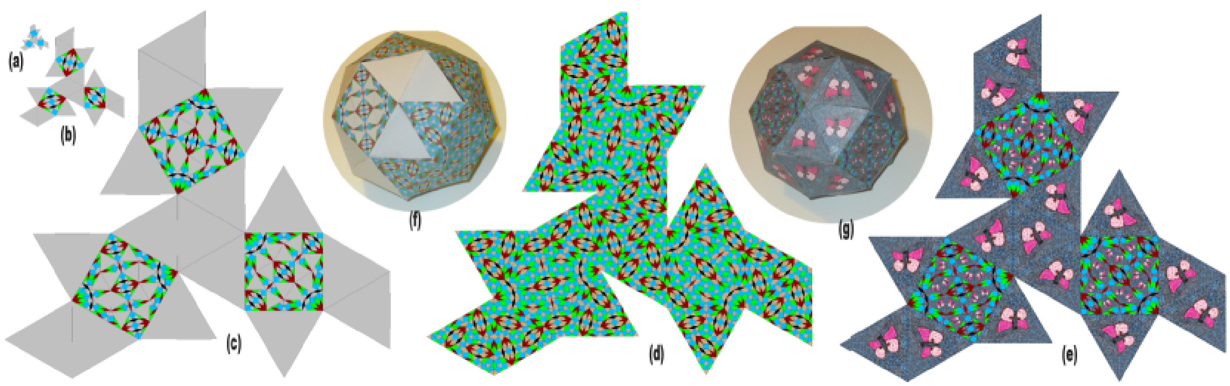

Figure 13: *The development of a fractal from a snub cube using a 12-gon decorated with a butterfly from the Hodthorpe Well Dressing.*

Name	triangles	squares	pentagons	hexagons	octagons	decagons	LCM
truncated tetrahedron	4			4			6-gon
cuboctahedron	8	6					12-gon
truncated cube	8				6		24-gon
truncated octahedron		6			8		12-gon
rhombicuboctahedron	8	18					12-gon
truncated rhombicuboctahedron		12		8	6		24-gon
snub cube	32	6					12-gon
icosidodecahedron	20		12				15-gon
truncated dodecahedron	20					12	30-gon
truncated icosahedron			12	20			30-gon
rhombicosidodecahedron	20	30	12				60-gon
truncatedicosidodecahedron		30		20		12	60-gon
snub dodecahedron	80		12				15-gon
Platonic solids	20-	6	12				60-gon

Figure 14: *The lowest common multiple for each solid shape.*

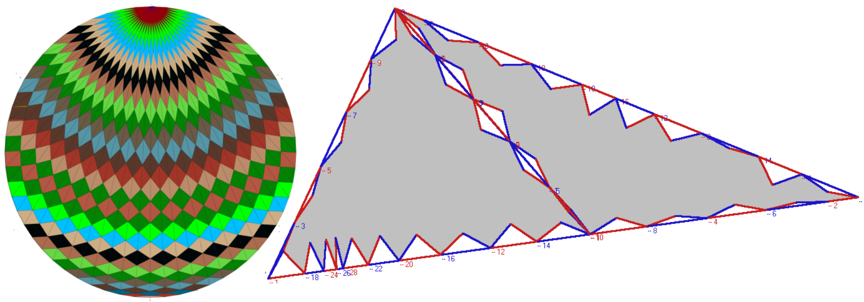

Figure 15 : *The development of a general approch for the 60-gon.*

required to complete the Archimedean and the five Platonic solids. I am developing a systematic approach for this problem which is required for n-gons with $n > 30$. I have proved that enlarged versions of the primary triangles can always be produced, including the 60-gon. For the 60-gon there are 29 primary triangles. I have also proved that the solution for 7 triangles will lead to a solution for all the 29 triangles. The complete filling of the triangles with primary triangles is not necessay. It is sufficient for the triangles that fit around the perimeter. See Figure 15. Each enlarged side is always the length of one or more diagonals.

The Software

The software routines were written Visual Basic. It could handle large graphic files, up to 100 megapixels. It would allow the development of a small picture, which takes a relatively short time. The final version, typically 50 megapixels could take several hours.

References

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[6] Rod Kirkpatrick video. Available online at url https://www.youtube.com/watch?v=jSvIf7OSmTU (as of April 20 2017)