Obtaining the H and T Honeycomb from a Cross-Section of the 16-cell Honeycomb

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Abstract

The two polyhedra H (Hexagonal bipyramid imaginary cube) and T (Triangular antiprismoid imaginary cube) form a honeycomb (3D tiling) which is obtained by truncating vertices of the cubic honeycomb, and at the same time, is obtained by truncating vertices of a honeycomb of triangular prisms. In this paper, we show that this honeycomb is obtained by shrinking a cross-section of the 4-dimensional honeycomb of 16-cells.

Introduction

An imaginary cube is a three-dimensional object which has square projections in three orthogonal ways just as a cube has [5]. A regular tetrahedron is an example of an imaginary cube (Figure 1(a)). In addition, there are two imaginary cubes with remarkable geometric properties; a hexagonal bipyramid imaginary cube (Figure 1(b), which we simply call H) and a triangular antiprismoid imaginary cube (Figure 1(c), which we call T). H and T form a honeycomb of the three-dimensional space as Figure 2 shows. This tiling was first presented in [2] and investigated in [5] with its application to a puzzle [4].

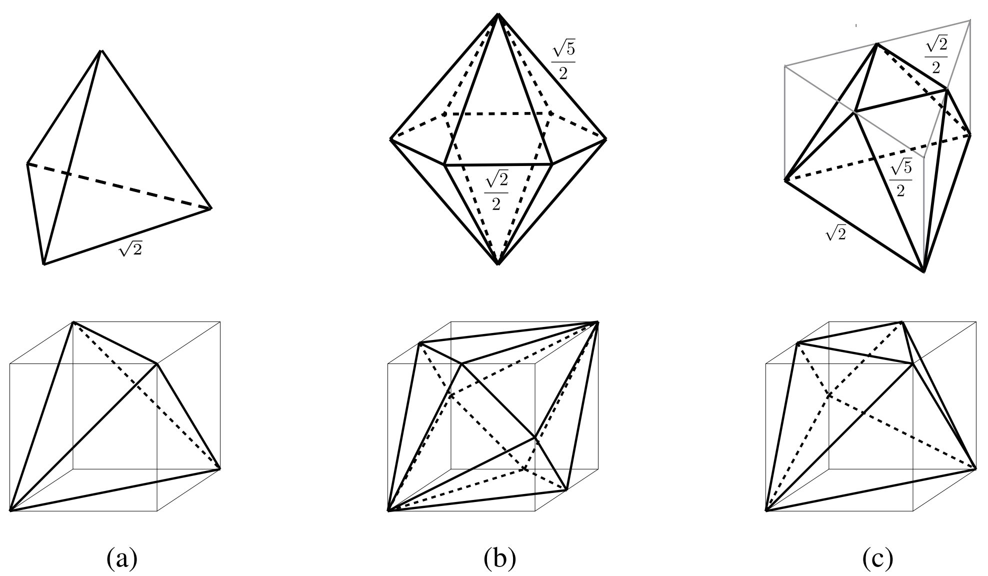

Figure 1 : (a) *a regular tetrahedron;* (b) *a hexagonal bipyramid imaginary cube (H); and* (c) *a triangular antiprismoid imaginary cube (T). The lower figures show how these polyhedra can be put into unit cubes so that they have the same three square projections as the cubes.*

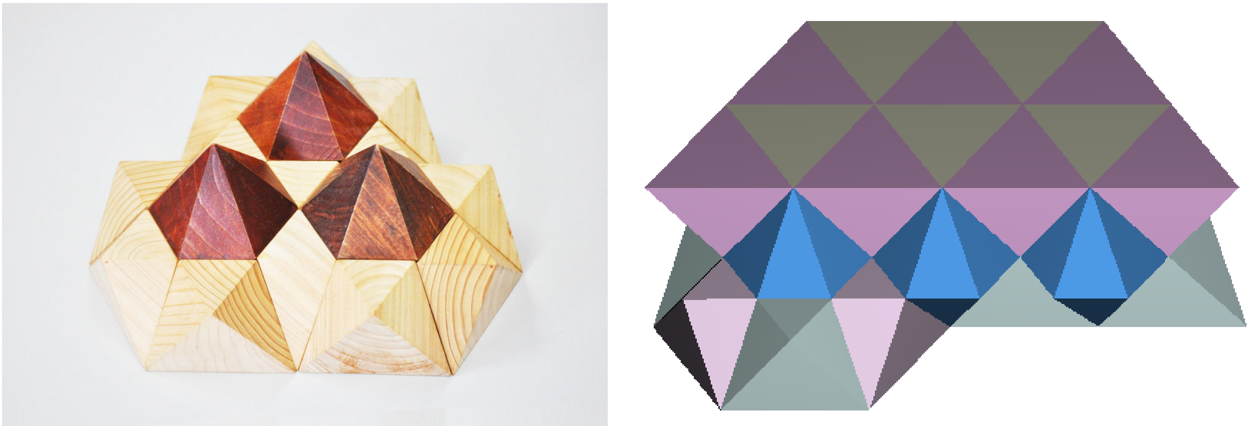

Figure 2 : *Parts of the H and T honeycomb.*

In [5], the author explained this honeycomb as a tiling of imaginary cubes and weak cross-polytopes, and also as a Voronoi Tiling. In this paper, we explain this honeycomb through its relation to the 16-cell honeycomb, which is one of the most fundamental honeycombs in 4-dimensional space. It is known that the tetrahedral-octahedral honeycomb (the honeycomb of regular tetrahedra and regular octahedra) and the rectified cubic honeycomb (the honeycomb of cuboctahedra and regular octahedra), which are well-known three-dimensional honeycombs by two semi-regular polyhedra, are obtained as cross-sections of the 16-cell honeycomb. We show that this honeycomb is "almost" a cross-section of the 16-cell honeycomb. More precisely, it is obtained from the 16-cell honeycomb by taking a cross-section with a hyperplane and then shrinking it in one direction by a factor of $\frac{\sqrt{3}}{2}$.

In the next section, we explain fundamental properties of the imaginary cubes H and T. Then, we review the explanations of this tiling as a tiling of imaginary cubes and weak cross-polytopes. After that, we introduce the 16-cell honeycomb and show a relation that exists between these two honeycombs.

Imaginary Cubes H and T

H is a hexagonal bipyramid which has isosceles triangle sides with the ratio of edges $(\sqrt{2}, \sqrt{5}, \sqrt{5})$ (Figure 1(b)). It fits into a cube so that it has the same three square projections as the cube, and therefore is an imaginary cube. Moreover, it is a double imaginary cube in that it is an imaginary cube of two different cubes.

T is a triangular antiprismoid. It is obtained by truncating three vertices of a triangular prism with the ratio of a bottom edge and the height $\sqrt{2} : \frac{\sqrt{3}}{2}$ (Figure 1(c)). It also fits into a cube and is also an imaginary cube. In addition, this octahedron has the property that all the six vertices can be arranged on the three coordinate axes (Figure 3). There are two distances from the origin to vertices and their ratio is 2:1. The longer distance is the same as the side-length of the cube in which it fits. Note that if all the distances from the origin to the vertices are the same, then it is a regular octahedron. We call a convex polytope all of whose vertices can be arranged on the axes of coordinates a weak cross-polytope and if, in addition, the distances from the origin to the vertices are the same, we call it a cross-polytope.

We will explain the H and T honeycomb through these properties of H and T. In Figure 1, some vertices of these polyhedra are also vertices of the cubes and others are on the midpoints of edges. We will call the former V-vertices and the latter E-vertices. A regular tetrahedron has four V-vertices which are alternating vertices of a cube. H has two V-vertices which are space-diagonal vertices of a cube and T has three V-

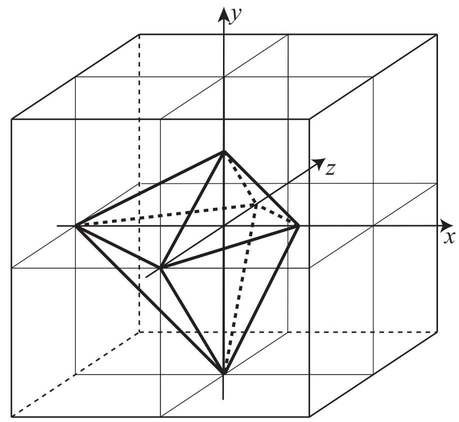

Figure 3 : *T as a weak cross-polytope.*

vertices which form an equilateral triangle. E-vertices are located at the middle points of the edges which do not contain V-vertices. Therefore, in each of these three imaginary cubes, every edge contains exactly one vertex and the arrangement of the V-vertices determine the location of the E-vertices.

Now, let us consider the subset $\{(x, y, z) \mid x + y + z \equiv 0 \pmod 2\}$ of the cubic lattice (Figure 4). Then, in each cube of this lattice, vertices coming from this set form a regular tetrahedron. The holes left in the space are regular octahedra (i.e., cross-polytopes) with the centers lattice points on $x + y + z \equiv 1 \pmod 2$. In this way, we have the honeycomb of regular tetrahedron and regular octahedron (Figure 6).

Next, consider the subset $\{(x, y, z) \mid x + y + z \equiv 0 \pmod 3\}$ of the cubic lattice (Figure 5). Then, in each cube of this lattice, we obtain H or T by selecting vertices in this set as V-vertices. The condition of the E-vertices that they are middle points of the edges without V-vertices ensures that polyhedra in adjacent cubes share the same vertices. Therefore, the holes left in the space are polyhedra with vertices on the lattice edges. They are weak cross-polytopes with the origins lattice points on $x + y + z \equiv 1 \pmod 3$ and $x + y + z \equiv 2 \pmod 3$. They are actually T. In this way, we have the H and T honeycomb, in which the ratio of the number of H to the number of T is 1:4 (Figure 7).

The 16-cell Honeycomb

A 16-cell is a four-dimensional regular polytope with 16 regular tetrahedral facets. One can consider it as a four-dimensional counterpart of a regular octahedron in that it is a 4-dimensional cross-polytope. Actually, $\{(\pm1, 0, 0, 0), (0, \pm1, 0, 0), (0, 0, \pm1, 0), (0, 0, 0, \pm1)\}$ is a set of vertices of a 16-cell.

A 16-cell can also be considered as a generalization of a regular tetrahedron; from the 16 vertices of a hypercube, one can take alternating vertices to form a vertices-set of a 16-cell. This observation shows that 16-cells are imaginary hypercubes, which are four-dimensional counterparts of imaginary cubes. In fact, they are double imaginary hypercubes. For more about imaginary hypercubes, see [6]. For more about four-dimensional geometry, see [1].

In this way, one can form from a 4-dimensional cubic lattice two kinds of 16-cells: one in each hypercube and one on each vertex of the lattice. Note that both kinds of 16-cells have the same size. Now, we take the construction similar to that of tetrahedral-octahedral honeycomb in the previous section. Take a subset $A = \{(x, y, z, w) \mid x + y + z + w \equiv 0 \pmod 2\}$ of the 4-dimensional cubic lattice. Then, in each hypercube, vertices that belong to A are alternating vertices and they form a 16-cell. We also have a 16-cell around lattice points $\{(x, y, z, w) \mid x + y + z + w \equiv 1 \pmod 2\}$, whose vertices are in A. They are space-filling and thus we have the 16-cell honeycomb.

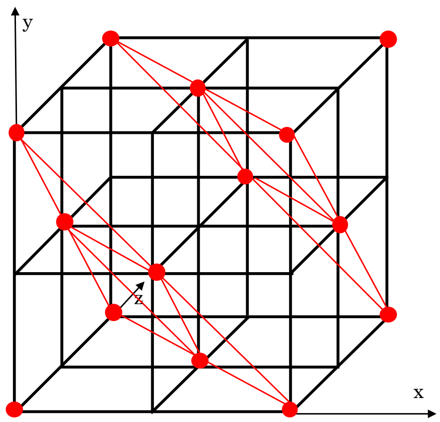

Figure 4: $x + y + z \equiv 0 \pmod 2$ *of the* $2\times2\times2$ *lattice; V-vertices of regular tetrahedra.*

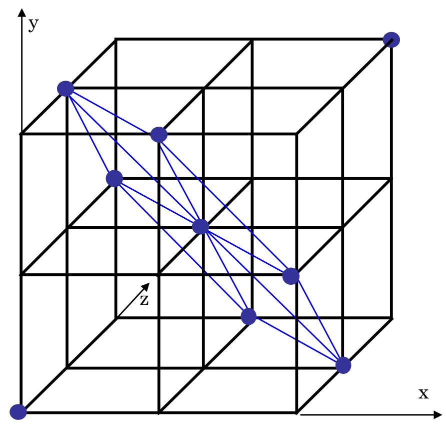

Figure 5: $x + y + z \equiv 0 \pmod 3$ *of the* $2 \times 2 \times 2$ *lattice; V-vertices of H and T.*

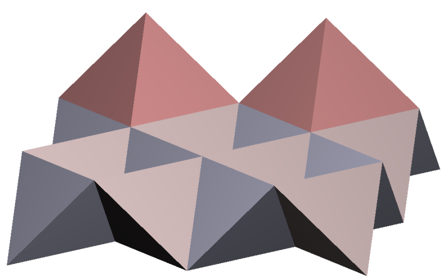

Figure 6: *The tetrahedral-octahedral honeycomb* ($4 \times 4 \times 1$ *lattice of tetrahedra and two octahedra).*

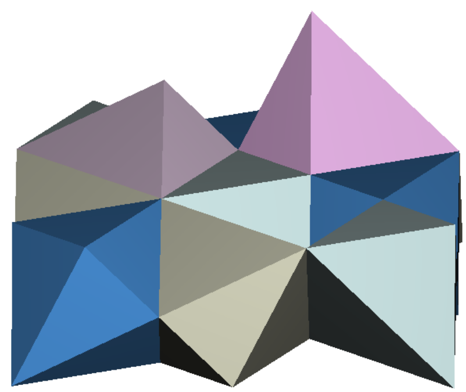

Figure 7: *The H and T honeycomb* ($3 \times 3 \times 1$ *lattice of H and T on imaginary cube positions and two copies of T on weak cross-polytope positions). Figure 2 is obtained by rotating it so that* $x = y = z$ *comes to the vertical axis.*

A Cross-Section of the 16-cell Honeycomb.

In general, one can obtain a 3-dimensional honeycomb by taking a cross-section of a 4-dimensional honeycomb with a hyperplane. From the 16-cell honeycomb, one can obtain the tetrahedral-octahedral honeycomb by taking a cross-section with a hyperplane like $x = n$ or $x + y + z + w = 2n$ for an integer n. One can as well obtain the rectified cubic honeycomb by taking a cross-section with a hyperplane like $x = n + 1/2$ or $x + y + z + w = 2n + 1$ for an integer n.

In this article, we consider a cross-section of the 16-cell honeycomb with $x + y + z = n$ for an integer n. As a representative, we consider the case $n = 1$. Note that $x + y + z = 1$ is mapped by the translation $(-1, 0, 0, -1)$ to $x + y + z = 0$ and this translation preserves the set A and thus the 16-cell honeycomb is preserved by this translation. Therefore, we have the same cross-section for every n.

First, consider the cross-section of the unit hypercube $[0, 1]^4$ with $x + y + z = 1$. It is the product of (the cross-section of the unit cube $[0, 1]^3$ with the plane $x + y + z = 1$) and the unit interval $[0, 1]$, which is $0 \le w \le 1$. Since the former is an equilateral triangle, this product is a triangular prism. Similarly, the cross-section of each hypercube with $x + y + z = 1$ is a triangular prism. Thus, the cross-section of the

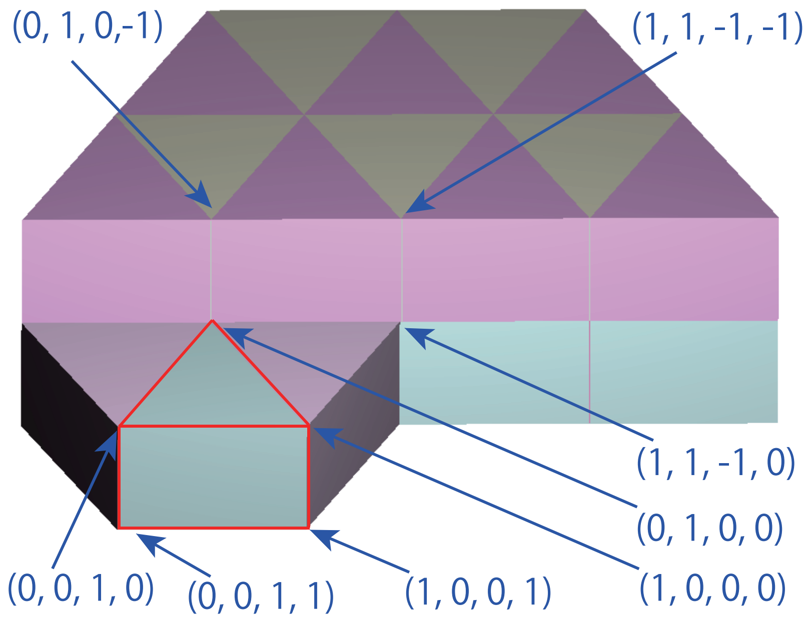

Figure 8: *A honeycomb of triangular prisms obtained as the cross-section of the hypercube honeycomb with $x + y + z = 1$. The edges of the prism obtained from the unit cube $[0, 1]^4$ are colored in red.*

four-dimensional cubic honeycomb is a honeycomb of triangular prisms in Figure 8.

Now, we consider the cross-section of the 16-cell honeycomb with the hyperplane $x+y+z = 1$. As we observed, the 16-cell honeycomb is composed of two kinds of 16-cells: 16-cells in hypercubes and 16-cells around the vertices $\{(x, y, z, w) \mid x + y + z + w \equiv 1 \pmod 2\}$.

The cross-section of the former kind of 16-cell has to be contained in a triangular prism. Let T be the 16-cell in the unit hypercube $[0, 1]^4$ and S be the cross-section of T. The set of vertices of T is $\{(0, 0, 0, 0), (1, 1, 0, 0), (0, 1, 1, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 1)\}$. Among them, (1, 0, 0, 1), (0, 1, 0, 1) and (0, 0, 1, 1) are contained in $x + y + z = 1$ and therefore they are vertices of S. For the other vertices, (0, 0, 0, 0) is in the half-space $x + y + z < 1$ and (1, 1, 0, 0), (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 1, 1) are in the half-space $x + y + z > 1$. Since (1, 1, 0, 0), (0, 1, 1, 0), and (1, 0, 1, 0) are connected with (0, 0, 0, 0) by edges, there are three more vertices of S, which are (1/2, 1/2, 0, 0), (0, 1/2, 1/2, 0), and (1/2, 0, 1/2, 0). Since no other edge of the 16-cell intersects the plane x + y + z = 1, S is a polyhedron with these six vertices, which is a triangular antiprismoid.

Note that, modulo translation, there are two ways that a hypercube $(x, y, z, w) + [0, 1]^4$ $(x, y, z, w \in \mathbb{Z})$ contains a 16-cell, just as there are two ways that a cube contains a tetrahedron, and there are two ways a hypercube is truncated by the hyperplane $x + y + z = 1$, just as there are two ways that a cube in a cubic lattice is truncated by the plane $x + y + z = 1$. Therefore, there are four kinds of cross-sections modulo translations. They are congruent (but differently oriented) triangular antiprismoids, as Figure 9 shows.

The cross-sections of the latter kind of 16-cells should be the spaces left around the vertices $\{(x, y, z, w) \mid x + y + z + w \equiv 1 \pmod 2\}$ in the honeycomb of triangular prisms. They are hexagonal bipyramids, as Figure 9 shows.

Thus, the cross-section of the 16-cell honeycomb with the hyperplane $x + y + z = 1$ is a honeycomb of triangular antiprismoids and hexagonal bipyramids. Note that these triangular antiprismoids and T have different heights and it is also the case for hexagonal bipyramids and H. The H and T honeycomb is obtained by shrinking this honeycomb obtained as a cross-section along the w-axis with the ratio $\frac{\sqrt{3}}{2}$.

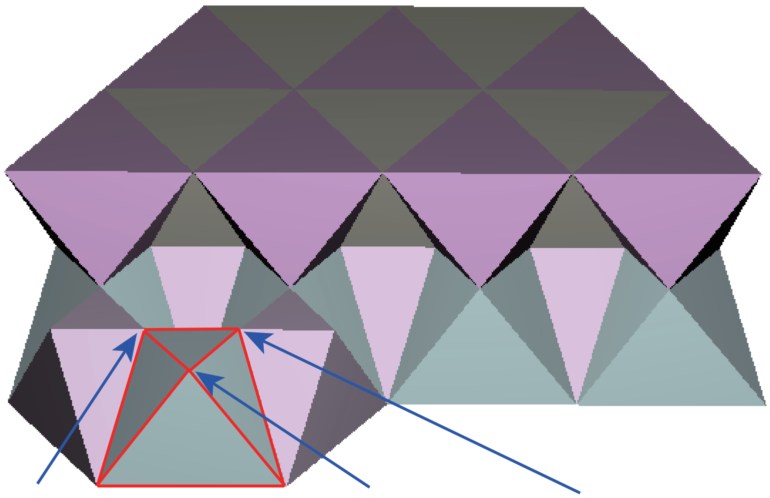

Figure 9 : *The cross-sections of the 16-cells in hypercubes with* $x + y + z = 1$. *Please compare this figure with the right figure of Figure 2.*

Concluding Remarks

In this article, we showed a connection between two imaginary (hyper)cube honeycombs: the H and T honeycomb and the 16-cell honeycomb.

H, T, and the regular tetrahedron are imaginary cubes with remarkable properties; H is a double imaginary cube, T is a weak cross-polytope, and the regular tetrahedron is a demicube (alternation of a cube). A 16-cell is an imaginary hypercube with all these three properties; it is a double imaginary hypercube, it is a cross-polytope, and it is a demihypercube. In addition, it is shown in [6] that T and the 16-cell are the only weak cross-polytope imaginary n-cubes for $n \geq 3$, and H and the 16-cell are the only double imaginary n-cubes for $n \geq 3$ under some conditions. Since these two honeycombs of imaginary (hyper)cubes are based on these properties, one can see a strong connection between these two honeycombs. In this article, we showed yet another connection between them.

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