Natural Color Symmetry

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Abstract

Existing techniques to create images with color-reversing (or other kinds of color symmetry) produce results that look less than natural. We produce images with *approximate* color symmetries by using the domain coloring algorithm with suitable complex Fourier series and photographs that have varying degrees of naturally occurring color symmetry. The technique yields patterns where the color symmetry looks less crystallographic than those previously produced, but perhaps more artistic. The technique produces an interesting combination of symmetrical and natural patterns.

Introduction

In *Creating Symmetry* [2], I expanded on a Fourier series method [4] to produce a comprehensive set of recipes for functions $f : \mathbb{C} \to \mathbb{C}$ with translational symmetry in two directions that *also* satisfy a *color-reversing* condition

$$f(\alpha(z)) = -f(z), \tag{1}$$

where α is some isometry of the plane. Such a transformation α is called a *color-reversing symmetry* of the pattern obtained from the function f. For instance, the right-hand side of Figure 1 shows a pattern with color-reversing quarter turns.

Figure 1 : *A color-reversing pattern (right) created from complex waves and a photograph of an African violet and its negative (left).*

Patterns like this one are constructed from photographs using the *domain-coloring algorithm* [3]. We recall this two-step process for depicting a complex-valued function $f(z)$, as follows:

1. Assign a color to each point of the complex plane, perhaps using a photograph.
2. Color pixels in the domain (or a subset thereof) of $f(z)$, by using the color $f(z)$ at the point z.

To depict a color-reversing patterns, I devised photographic collages, where the color of $-z$ is the photographic negative of the color of z. For instance, on the left in Figure 1, a photograph of African violets is juxtaposed with a rotated negative of the same photograph in order to accomplish this negative property. When such a color wheel is used in conjunction with a function that satisfies (1), the result is an image that depicts the color-reversing property very accurately, as on the right in Figure 1. Since the actual symmetry group of the pattern is cmm and the group obtained by lumping together all symmetries and color-reversing symmetries is p4g, we call this pattern type p4g/cmm.

In this paper, I question whether patterns created from such a surgically created color wheel lead to the most artistically satisfying patterns that might be possible with the technique of domain coloring. Although the pattern in Figure 1 is arguably more interesting than the purely diagrammatic ones from well-known mathematical sources [5, 9], I propose three other methods that may be even better.

For instance, the photograph on the left in Figure 2 shows the stump of a red fir tree that had been cut in such a way to create two differently-colored halves. By positioning the photograph carefully in the complex plane, locating the origin of the complex plane on the line where the image changes so distinctly, we create a color wheel that has some of the character of a color-reversing wheel, without taking the instructions too literally. Using *exactly* the same color-reversing complex waves that led to the pattern in Figure 1, but with the stump photograph as color wheel, we find the pattern in Figure 2. We argue that this combination of symmetry and approximate symmetry leads to more aesthetically pleasing patterns than the precisely rigged color wheels.

Figure 2: *An image with approximate color-reversing symmetry created from* the same *complex waves and a photograph of a freshly cut fir stump.*

Color-Reversing Recipes

For any desired wallpaper group acting on the plane, we can work out complex-valued Fourier series to capture every possible function invariant under that particular group. For instance, if we would like to create patterns of type p2 with a generic lattice spanned by complex vectors 1 and $a + bi$, we start with the elementary integer-frequency waves

$$E_{n,m}(X, Y) = e^{2\pi i(nX+mY)}, \text{ where } X = x - ay/b \text{ and } Y = y/b. \tag{2}$$

A linear combination of these waves will be periodic with respect to the right lattice, but we only achieve p2 symmety if we follow the recipe

$$f(x, y) = \sum_{n,m\in\mathbb{Z}} a_{n,m}E_{n,m}(X, Y), \text{ where } a_{n,m} = a_{-n,-m}. \tag{3}$$

We can check that keeping the given coefficients locked together according to the recipe produces a function invariant under the full group [2].

Here's one example to show how to create color-reversing symmetry. If, in addition to the recipe above, we choose our coefficients so that every nonzero term has frequencies with $n + m$ odd, then the function acquires a *negating half-turn*. This means that

$$f(-X + 1/2, -Y + 1/2) = -f(X, Y).$$

To check this, compute that the given transformation takes each of the component waves in (2) to the negative of the wave that it is locked with. Hence, we get color-reversing symmetry.

Suppose we have a pattern made from a function f with p2 symmetry that satisfies this color-reversing condition as well. The symmetry group of $|f|$ includes the additional half-turn and can be seen to be a different representation of the same group, p2 (with smaller translational symmetries). We call this larger group, which includes symmetries together with color-reversion ones, the *color group* of the pattern. Color-reversing pattern types are classified by a symbol "color group/symmetry group," so a pattern made from the recipe here would be classified as having type p2/p2. (Many readers will know the work of Washburn [8], who advised museums to classify patterns according to mathematicians' crystallographic schemes.)

The color-reversing nature of a pattern made from a function that follows this p2/p2 recipe will only *reveal* its color-reversing nature if we color the plane carefully. We need to be able to detect the complex value $f(z)$ as being, in some sense, the opposite of $-f(z)$. In past work, I universally colored such functions with a collage like the color wheel in Figure 1, where a photograph is juxtaposed with an upside-down copy of its negative. As noted in the introduction, this often leads to sharp edges and images that are perhaps a little too cut-and-dried in their symmetry.

Let's explore what happens when we make different choices. There should still be some visual relationship between the points z and $-z$ in the color assignation used for Step 1 of the domain coloring algorithm, but it need not be one of strict photographic negation.

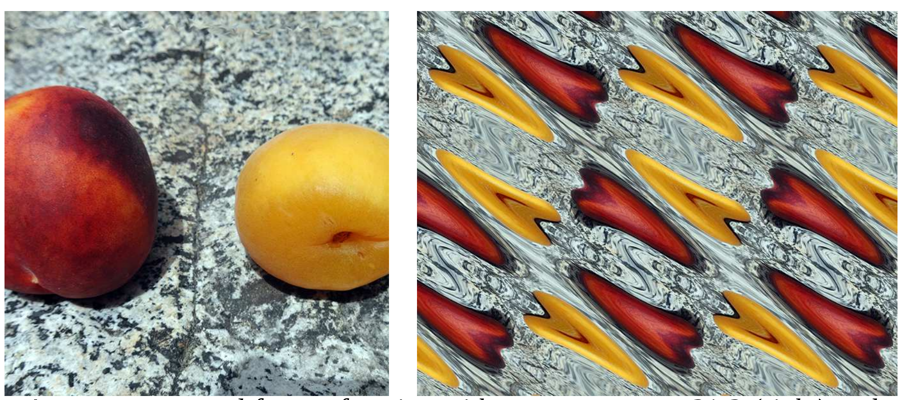

Figure 3: *An pattern created from a function with symmetry type c2/p2 (right), colored with a photograph that is only vaguely color-reversing when turned upside down (left).*

For instance, in Figure 3, turning the photograph upside down reverses the position of the peach and the apricot. The granite is relatively uniform, although the peach shadow creates a darker patch on one side. In the computed pattern, I see pairs of red and yellow hearts, with the yellow ones appearing to be approximately-rotated negatives of the red ones. The p2/p2 idea comes through, as long as we treat the granite as a neutral color that serves as its own opposite. Is it a success? It is only one of infinitely many possibilities for patterns with this symmetry type that use this particular naturally two-colored photograph. I like it, but continue to wonder if there were a better one nearby waiting to be found.

We've shown examples for only two of the possible 46 color-reversing pattern types. Instead of exhausting the types, let's move on to other ideas for natural symmetry.

Conjugate Symmetry

The mapping $z \to -z$ is only one of two nice maps of the complex plane to itself that can intertwine with order 2 elements of wallpaper groups. The other is conjugation: $z \to \overline{z}$, a flip about the x-axis. We say that an isometry of the plane, α, is a *conjugate symmetry* of a function f from $\mathbb{C}$ to itself to mean that

$$f(\alpha(z)) = \overline{f(z)}.$$

Instead of negating the output value of f, such a transformation returns the conjugate.

The algebra of conjugate symmetries is extremely similar to that of color-reversing symmetries. For instance, the square of a conjugate symmetry is always a symmetry, and the set of symmetries and conjugate symmetries of a pattern form a group—which I'll call the *conjugate color group*—of which the symmetry group is a normal subgroup of index 2. The difference seems to be mostly in the Fourier series construction of functions to make patterns patterns. I'll offer one example here, first showing how to construct functions for a particular pattern type and then explaining the consequences for domain coloring.

Let's construct functions with cm symmetry and then find conjugate symmetries to create patterns with conjugate color group cmm. The relevant wave functions are

$$E_{n,m}(X,Y) = e^{2\pi i(nX+mY)}, \text{ where } X = x + iy/b, Y = x - iy/b, b \in \mathbb{R}, \text{ and } n, m \in \mathbb{Z}.$$

Evidently, reflection across the x-axis in the domain of this function swaps the waves with indices (n,m) and (m,n), so we find cm functions by choosing series where the coefficients satisfy

$$a_{n,m} = a_{m,n}.$$

To achieve a conjugate color group cmm, we need reflection across the y-axis, which I'll call σ_y, to be a conjugate symmetry. With f just as in (3), but with the X, Y, and Es defined differently, we compute

$$f(\sigma_y(z)) = f(-\overline{z}) = \sum_{a,m} a_{n,m} E_{n,m}(-Y,-X) = \sum_{a,m} a_{n,m} E_{-m,-n}(X,Y) = \sum_{a,m} a_{-m,-n} E_{n,m}(X,Y),$$

where we just re-indexed in the last step.

On the other hand,

$$\overline{f(z)} = \sum_{a,m} \overline{a_{n,m}}\,\overline{E_{n,m}(X,Y)} = \sum_{a,m} \overline{a_{n,m}} E_{-n,-m}(X,Y) = \sum_{a,m} \overline{a_{-n,-m}} E_{n,m}(X,Y).$$

We have already agreed to have $a_{n,m} = a_{m,n}$ for our function, so the conjugate symmetry condition is true if and only if the coefficients $a_{n,m}$ are all real! That's an easy choice, so implementing this recipe for conjugate symmetry required no additional programming to lock waves together.

Now that we know how to find functions with symmetry group cm and conjugate symmetry group cmm, let's consider how to picture them. For color-reversing symmetry, we would like a photograph where the color at the point z is the negative of the color at $\overline{z}$. That's quite easy to arrange with a photographic collage, but the effect is no more natural than what we achieved with color-reversing symmetry. Instead, we look for photographs where there is only approximate negation.

Before doing that, let's pause to see what happens when we use a photograph that is approximately *the same*, rather than opposite, when you flip it across a horizontal axis. I took a rather symmetric picture of a symmetric building in a symmetric setting: the Rosicrucian Park in my neighborhood. I had to rotate it so that the existing left-right approximate symmetry would line up with the x-axis, though it is displayed

 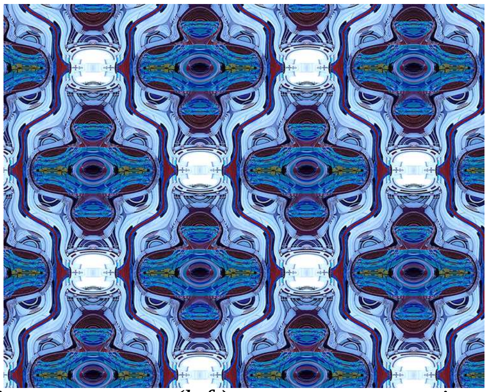

Figure 4 : *A photograph with approximate mirror symmetry (left) creates pattern using a function with conjugate symmetry type cmm/cm. Note: I rotated the photograph* $90°$ *before using domain coloring.*

here to promote viewing from a familiar angle. Using this with the any of the functions we just constructed should yield symmetry of type cm, but only *approximate symmetry* of type cmm.

The source photograph is not the only thing that has been turned; the computed image has been turned as well, for artistic effect. Careful examination will show that the mirror reflections across vertical axes are precise, while those across horizontal axes are only so-so. It's fun to find the differences made by the tree against the sky on one side, as well as the variation in the windows. In the original image, the cross in the tiled foreground appears to be centered, but the computed image shows that it is not perfectly so.

From an artistic point of view, I find this image more pleasing than many cmm ones I have made. The near miss of symmetry in the original photograph brings an interesting extra variation in the computed image. It seems to me that our minds most enjoy a balance of symmetry and asymmetry.

To show approximate color-reversing conjugate symmetry, I constructed waves with symmetry type p31m and colored them with a minimalist photograph with only a vague gesture toward conjugate symmetry. Analysis quite similar to what I did above for the cmm/cm pattern will show that simply taking real coefficients in a sum of waves with p31m symmetry will turn on an extra mirror symmetry to create a p6m pattern. When that new mirror is colored in a reversing (or very approximately reversing) source photo, we get a pattern of type p6m/p31m. My example appears on the right in Figure 5.

 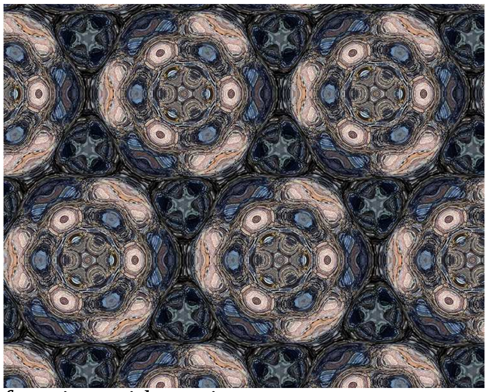

Figure 5 : *A pattern (right) created from a function with conjugate symmetry type p6m/p31m, colored with a photograph where color-reversing conjugate symmetry is particularly vague (left).*

Three-Color Symmetry

The types of 3-color symmetry in wallpaper are numerous [1, 2]. Here let's recap a couple of easy possibilities. As with every wallpaper group acting on the plane, the groups with 3-fold symmetry have their own spaces of waves from which we can select invariant functions. One way to achieve 3-color symmetry is to lock these waves together in packets to create a function f with *color-turning* symmetry, by which I mean

$$f(\omega_3 z) = \omega_3 f(z), \text{ where } \omega_3 = \frac{1+\sqrt{3}i}{2}. \tag{4}$$

This means that a turn of $120°$ about the origin in the domain of the function produces a multiplication of the output value by this cube root of unity: a turn of $120°$ in our color space.

The easiest recipes to implement involve congruence classes. Waves, as in the other examples in this paper, are indexed by pairs of integers (n, m). If we choose frequencies in every term of our sum to satisfy $n - m \equiv 1 \pmod 3$, then the function will have the color-turning symmetry in (4). All such waves necessarily pick up a *color-turning third-translation*, which I will define simply by example. If the generating translations in the group are by complex numbers 1 and ω_3, then the function satisfies

$$f\left(z + \frac{2+\omega_3}{3}\right) = \omega_3 f(z).$$

(Translation along the vector $2 + \omega_3$ travels the long diagonal of the hexagonal fundamental cell.)

I started with a p3 function and chose pairs satisfying this congruence condition to produce a function of color-turning type $\mathrm{p3}/_3\mathrm{p3}$, which means that the function f has p3 symmetry and the function $|f|$ has p3 symmetry, but with a different representation of the group with shorter translations are along the third of the main diagonal, as mentioned before.

Figure 6: *A composition of food slices has approximate color-turning symmetry, as does the digital image produced from color-turning waves.*

To color the function, I composed a source photograph from approximate third-circle slices of orange, tomato, and onion, with a blueberry in the middle. When you turn the photograph, the colors more-or-less turn. The resulting digital artwork appears in Figure 6b.

Note the amusing rhythm of "onion, tomato, orange" as you move upward along a line of slope $60°$; those are actual translational symmetries of the absolute value of the function that produced this pattern. If the color symmetry in the original photograph were exact, with a perfectly round blueberry at the center to indicate a neutral color that does not turn, then those shapes made from the blueberries would actually have 3-fold rotational symmetries, as they are centers of 3-fold rotation in the color group of the function that produced the computed image. Instead, they have a charming off-kilter quality. This is what I like about this natural approach to color symmetry: we see features that balance symmetry with wavy goofiness.

The example in Figure 7b uses the same congruence condition, but this time with waves having p3m1 symmetry. The resulting type is called $\mathrm{p31m}/_3\mathrm{p3m1}$ because the symmetries of the absolute value of this function include rotations about 3-centers without mirror axes through them.

Figure 7: *A plate of sweet peas of different hues (left) yields an approximately color-turning pattern (right).*

In this example, the 3-fold color symmetry of the original photograph is so weak that the would-be color symmetries along the main diagonal of the cell are extremely approximate. Still, the rhythm of this pattern is distinctive and would not be mistaken for a simple p3m1 pattern that had nothing else to recommend it.

Conclusion

If I have given you ideas for your own experiments, you may enjoy some new open-source software available for just this purpose. In the summer of 2016, students Bridget Went and Son Ngo at Bowdoin College wrote a new graphic user interface [6] for my existing suite of wallpaper software. Funding came from Bowdoin through a grant entitled *SymmetryWorks!*, which funded an art exhibition and my visit to their campus in the fall of 2016.

One nice feature of the software is that you can call up a window (Figure 8) to show which pixels of your photograph are being called by the wallpaper function currently held in the program settings. This helps to steer the waves toward desirable features of the source image. It also raises an interesting mathematical question: Can we predict the range of a wallpaper function from its coefficients?

You can download the source code, which compiles through the QtCreator platform, get a zip file with everything you need to run the executable in Windows, or find the .dmg file for Macs.

Looking ahead to future work, I mention the image in Figure 9. The analytic techniques in this paper can be used with a source image contrived to consist of exactly three colors to produce new figurative tilings. Software like that described by Adanova and Tari [7] at Bridges 2016 could be used to discretize the tiling to allow laser-cutting or 3D printing. For another example, consider the pattern on the right in Figure 2, which shows a tiling of the plane by two different tiles. There may also be interest in combining these observations about natural color symmetry with tiling or kaleidoscope programs.

With this new technique for natural color symmetry, mathematics and nature both contribute to the appearance of symmetry in computed images. It seems to me that our human minds like both and enjoy seeing them play well together.

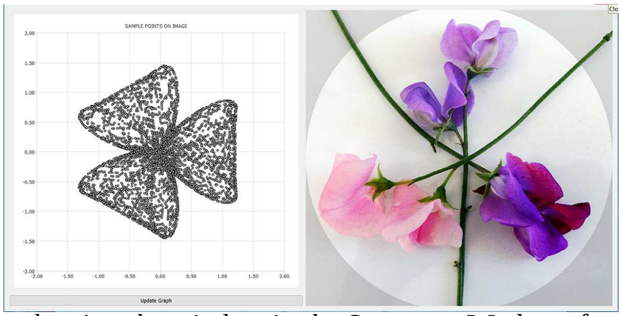

Figure 8 : *Screenshot showing the window in the* SymmetryWorks *software that allows you to see the range of the current wallpaper function alongside the source photograph. Here you can see that the function is calling for pixels in a perfectly symmetric way.*

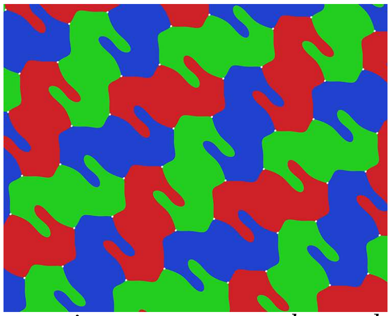

Figure 9 : *Functions with color-turning symmetry can be used to seek interesting tilings, as in this seahorse pattern made from a function of type* $\mathrm{pgg}/_3\mathrm{pg}$.

References

[1] John .H. Conway, Heidi Burgiel, and Chaim Goodman-Struass. *The Symmetries of Things*. A.K. Peters, 2008.

[2] Frank Farris. *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns*. Princeton University Press, 2015.

[3] Frank Farris, Review of *Visual Complex Analysis*, by Tristan Needham. Amer. Math. Monthly, 105: 570–576, 1998.

[4] Frank Farris. Symmetric yet organic: Fourier series as an artists tool. J. Math. Arts, 2013. doi: 10.1080/17513472.2013.819314

[5] Branko Grünbaum and G.C. Shephard. *Tilings and Patterns*. W. H. Freeman and Co., 1987.

[6] Son Ngo and Bridget Went. SymmetryWorks software. Available for download at `https://github.com/imrimt/symmetryworks-research-bowdoin` as of January 29, 2017.

[7] Venera Adanova and Sibel Tari. Extracting Unit Cells from Tilings with Color Symmetries: Case of Counterchange Patterns. In Carlo Sequin Douglas McKenna Kristof Fenyvesi Eve Torrence, Bruce Torrence and Reza Sarhangi, editors, *Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture*, pages 453–456, Phoenix, Arizona, 2016. Tessellations Publishing. Available online at `http://archive.bridgesmathart.org/2016/bridges2016-15.html`.

[8] Dorothy Washburn and Donald Crowe. *Symmetries of Culture, Theory and Practice of Plane Pattern Analysis*. University of Washington Press, 1988.

[9] H.J. Woods. The Geometrical Basis of Pattern Design Part IV: Counterchange Symmetry in Plane Patterns. Journal of the Textile Institute, Transactions, 27: 305–320, 1936.