Artwork Inspired by Dual Dodecahedra and Icosahedra

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Abstract

A long-standing collaboration between an artist and a mathematician bears new fruit in the form of novel geometric constructions and several works of art based on them. The inspiration for the present collaboration is a recent result on the edge-length ratios between dual regular dodecahedra and icosahedra, specifically the two vertex-to-face pairings of these dual Platonic solids: when the icosahedron circumscribes the dodecahedron, the edge-length ratio is $\phi/3$, and conversely, when the dodecahedron circumscribes the icosahedron, the edge-length ratio is $\phi^2/\sqrt{5}$, where ϕ is the golden number. These two edge-length ratios are the basis for the geometric constructions and the resulting artwork, which highlight interesting characteristics of the two ratios, both individually and as they relate to each other.

Introduction

Mark is an artist, researcher, and educator, with 20 years of experience teaching geometry and geometric systems, from an artist's point of view, to undergraduate and graduate students at the Academy of Art University in San Francisco. A review of Mark's second show at the Pierogi gallery (New York, June 2015) published in *The New Yorker* magazine acclaims "the exquisite drawings of this San Francisco-based artist, who, armed with straightedge and compass, transmutes the mysteries of geometry into dense meshes of colored lines, alive with spiritual intensity [1]."

Steve is a mathematics professor with publications in architectural history and the math/art interface, e.g., [2] and [6]. Steve's recent article in *The Fibonacci Quarterly* involving edge-length ratios between dual Platonic solids [5] is the inspiration of the present collaboration between him and Mark, which will require a little background to understand and appreciate fully.

Mark and Steve first met through the conference/journal *Nexus: Relationships Between Architecture and Mathematics*, and they have had a long collaborative relationship. For example, they published companion articles in the *Nexus Network Journal* extending the notion of arithmetic and geometric progressions to harmonic progressions, a question that arose from Mark's experiments with geometric relationships [3, 4]. A major thrust of Mark's work is to combine geometric systems that are from incommensurable ratios, analogous to music written in different keys without the advantages of equal temperament, yet sounding beautiful together. The common link between incommensurables may be a specific length of line, like a side or a diagonal, or the anatomy of a geometric construction.

A surprisingly new result. Steve and a former student at Sweet Briar College published for the first time [5] the exact values (as opposed to decimal approximations) of all five edge-length ratios between dual Platonic solids in vertex-to-face configurations (two for the cube and octahedron, two for the dodecahedron and icosahedron, and one for the self-dual tetrahedron). Figure 1 shows the two vertex-to-face configurations for a dodecahedron and an icosahedron; on the left the inner icosahedron has its vertices located at the centroids of the faces of the outer dodecahedron, and on the right the roles are reversed.

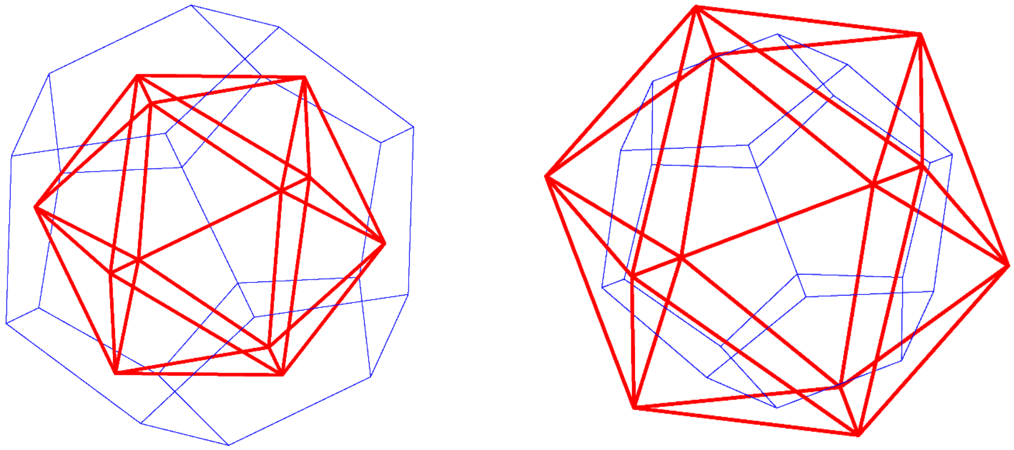


Figure 1: *The two vertex-to-face configurations of dual dodecahedra and icosahedra.*

At left in Figure 1, the ratio between the inner icosahedron's edge length and the outer dodecahedron's edge length is $\phi^2/\sqrt{5}$, whereas at right the edge-length ratio (again from inner to outer polyhedron) is $\phi/3$, where ϕ is the golden ratio. As surprising as it seems, the latter of these edge-length ratios seems to have been absent from the literature before [5]. Another surprising fact, though one that was already known, is that the former ratio is greater than 1, i.e., the edge length of the inner icosahedron at left is larger than the outer dodecahedron. This is the only one of the five edge-length ratios of vertex-to-face dual Platonic solids that is greater than 1 (when taking the ratio from inner polyhedron to outer).

The Present Collaboration. Knowing Mark's interest and experience in geometric constructions involving such ratios, Steve suggested that Mark experiment with $\phi^2/\sqrt{5} \approx 1.1708$ and $\phi/3 \approx 0.5393$. Much to the delight of both, Mark discovered some fantastic relationships, both exact and approximate. This article presents the highlights of these findings as well as a sampling of the works of art that Mark produced based on the results.

Exact Relationships

One fact that Mark discovered quickly is that the reciprocals of the ratios $\phi^2/\sqrt{5}$ and $\phi/3$ are, respectively, approximately 0.8541 and 1.8541. Steve verified mathematically that the relationship is exact, namely that $\sqrt{5}/\phi^2 + 1 = 3/\phi$. Mark knows decimal approximations for scores of ratios arising in his research and refers to them as such. Thus, as we shall see in the titles of his works below, Mark refers to the left ratio in Figure 1 as either "the 1.1708" or "the .8541" and the right ratio in Figure 1 as "the 1.8541"; whether he refers to a ratio or its reciprocal is a matter of choice, of course, and in any case it simply depends on whether the edge lengths in Figure 1 are divided inner to outer or outer to inner.

Exact geometric constructions. The first geometric construction included here, Figure 2, shows Mark initially discovering the fact cited above that $\sqrt{5}/\phi^2 + 1 = 3/\phi$, but this is actually an added bonus on top of the primary construction, a ϕ^2 by $\sqrt{5}$ rectangle as the union of 3 squares and 5 golden rectangles. There are squares and golden rectangles of two different sizes each. As Mark explains in the text above the construction, the three principal rectangles are composed of three other rectangles, as follows. The lower two squares, which are unit squares, and the golden rectangle between them comprises the ϕ^2 rectangle *AGNZ*. The lower right square is also part of the $\sqrt{5}$ rectangle, *RPMZ*, together with the golden rectangles above the square. The third rectangle is then the upper right region, *GKPS*, consisting of a different scale square and two different scale golden rectangles. Finally, note that the text in the upper right exhibits the geometric result of $\sqrt{5}/\phi^2 + 1 = 3/\phi$, namely that a square appended below *AKMZ* creates a $3/\phi$ rectangle, which caused Mark to comment in writing: "This is very cool!" Readers with little experience in such geometric constructions should understand that this sort of research is Mark's area of expertise (along with making artwork based on it), so hopefully they will trust his opinion on this delightful discovery.

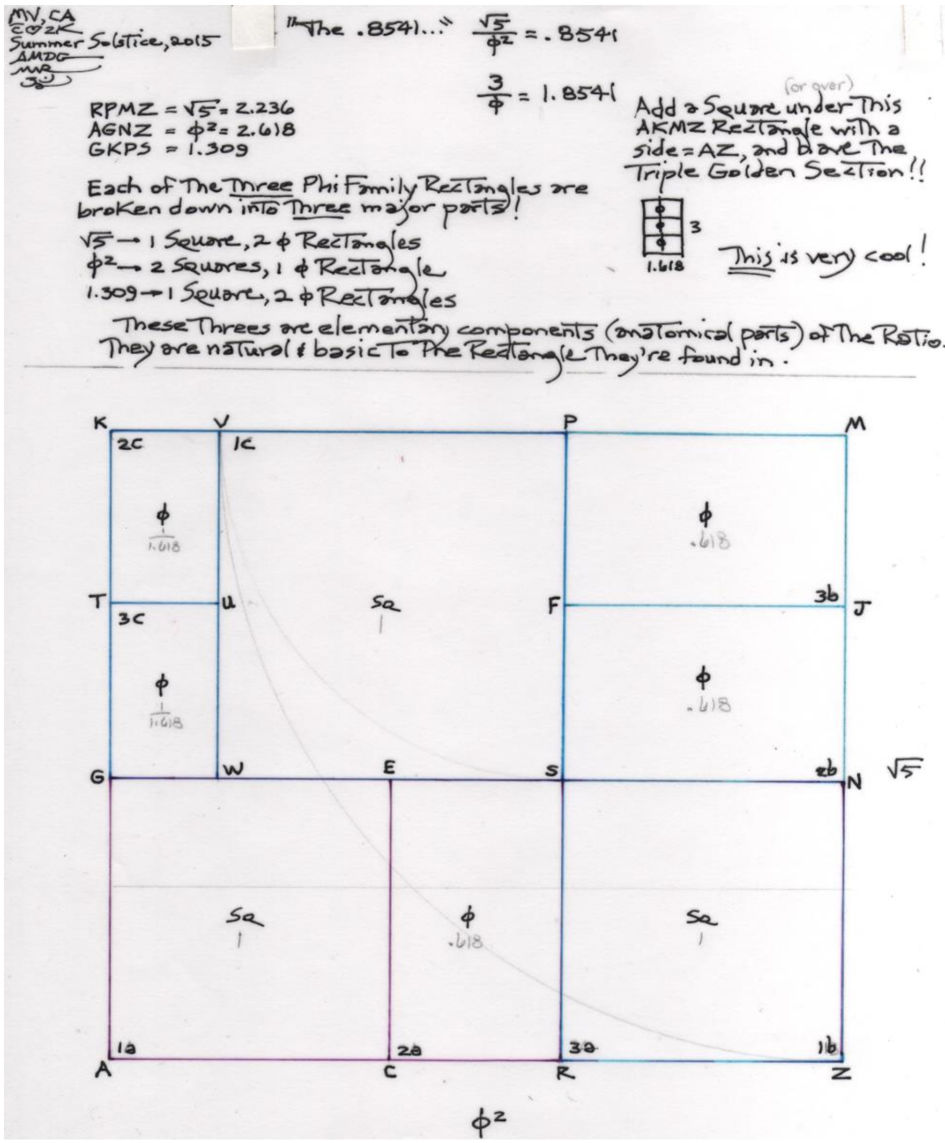


Figure 2: "The .8541…"

Figure 3 shows another exact construction. There are three golden rectangles stacked with portrait orientation on the left and three golden rectangles stacked with landscape orientation on the right, leaving a rectangle at the upper right. If we take $AL = 1$, then $LZ = \phi$, $ZE = 3$, $AK = 3\phi$, and $EM = 3\phi - 3 = 3(\phi - 1) = 3/\phi$ (of course, this works for any positive integer *n* in place of 3). The wonderful fact here is that not only can one visualize $3/\phi$ in Figure 3 by virtue of the ratios ZE/LZ and AK/AZ, but also one can measure $3/\phi$ by the length of the line segment EM (since $AL = 1$)! The other noteworthy part of the construction is that the upper right rectangle is composed of three stacked ϕ^2 rectangles.

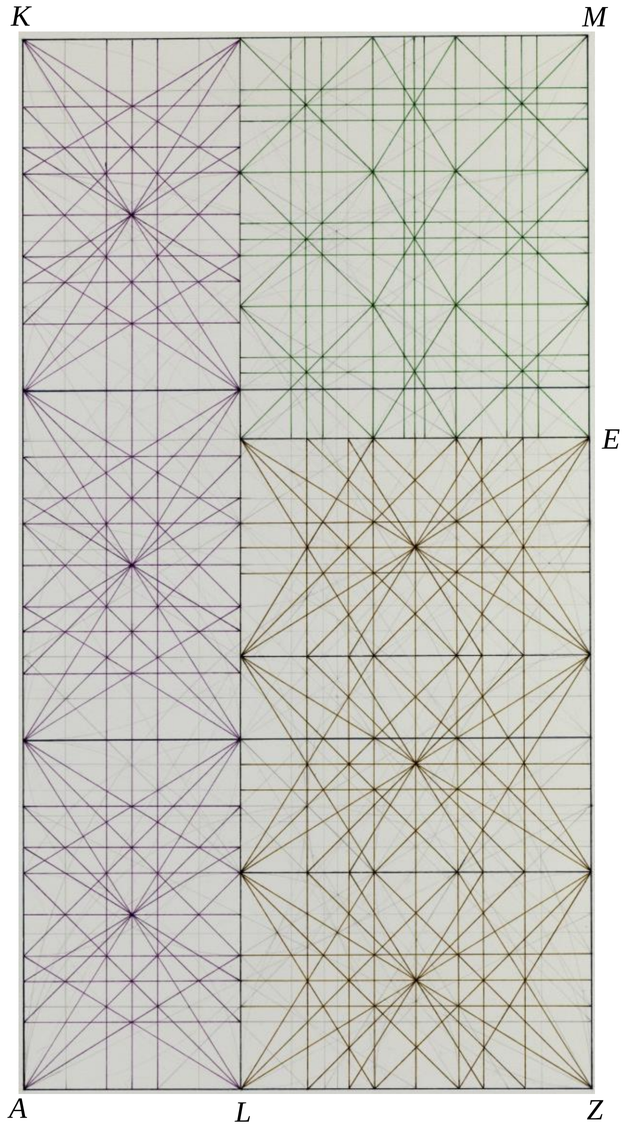


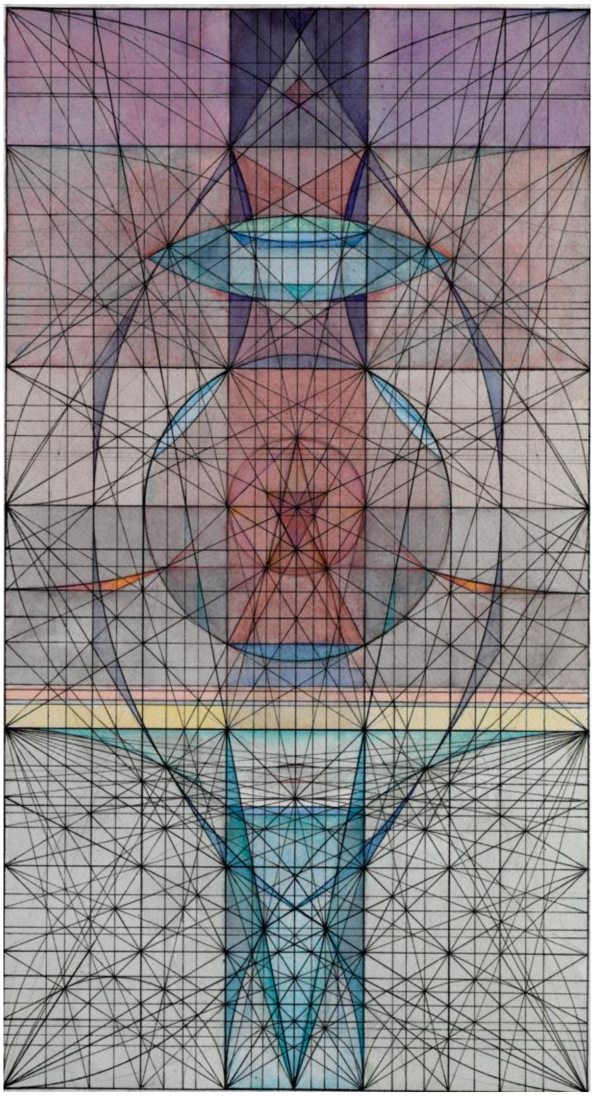
Figure 3: Asymmetrical Tiling in the 1.8541 *(the letters are added for reference).*

Marriages of Incommensurables

We move now from exact relationships to approximations. Recall that Mark likes to combine geometric systems that are from incommensurable ratios; in his words:

> My goal is to create harmony that resolves the initial numeric conflict, and my resolution is to draw the resulting compositional grid—a harmonic composition generated by the union of the two ratios, their shared unit, and the parts I select to join with straightedge and compass. The grid is unique to the marriage. I call them 'Marriages of Incommensurables', unions of ratios that cannot be measured together but can be constructed so that they work together. And, although the marriage is a vital component, the 'grid's the thing', for it is the grid that manifests the relationship originally worked out. I love making the grids, and having all the intersections coincide. For me, it's like making a map of night sky.

Figure 4 (at left) and Figure 5 (at right) are examples of finished works of Mark's using this approach.



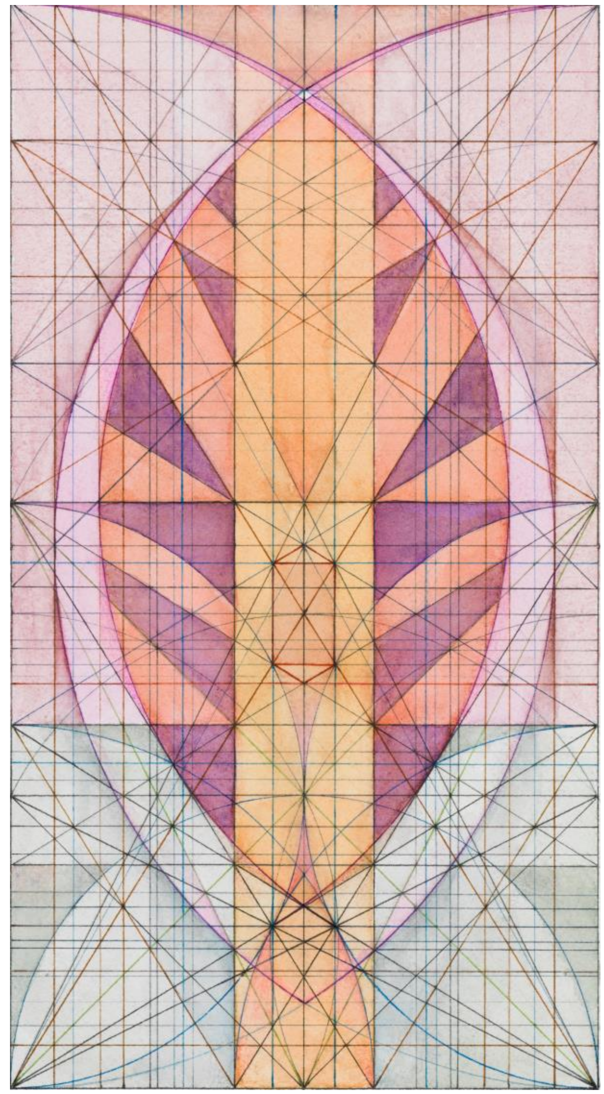


Figure 4: Alchemical Still with Alembic *(Watercolor and India ink on cotton paper, 24 in. x 18 in.)*.
Figure 5: Earth Tulpa *(Watercolor and colored inks on cotton paper, 14.125 in. x 10.25 in.)*.

At the core of Figures 4 and 5 is the geometric construction in Figure 6. The essentials of the construction are as follows. Starting with a unit square *ACEZ* bisected by *PR*, draw *PZ*. Swing an arc centered at *P* from *A* to *G*, and then swing an arc centered at *Z* from *G* to *L* (and up to *N*), in order to cut *AZ* in extreme and mean ratio, so that *AWNZ* is a golden rectangle. Stack two more copies of this golden rectangle to create rectangle *AKMZ* of proportion $3/\phi$, the 1.8541. Now swing an arc centered at *C* with radius *CG* up to point *K*. The rest of the construction can be done with symmetry. One can then see by symmetry (or by computation) that all intersections (including tangencies) of arcs and line segments in the construction are exact, except at *K* and *M*: while *CK* is 0.85410, *CG* is 0.85065 (recall *AC* = 1). Note that this is a difference of 0.00345, or taken as a percentage of *AK*, an error of about 0.2%.

Mark has repeated this construction a few times, and the correspondence between the extended arcs of the mandorla (a.k.a. vesica piscis) and the top of the rectangle at points *K* and *M* behaves as if the construction were exact. Mark points out that even exact constructions are never perfect when done with pencil and paper (or any other media), and this construction is so close that it is within the thickness of a pencil mark. A "marriage of incommensurables" often involves such imperfection, just as any marriage of humans does, but still the marriage can be very happy indeed!

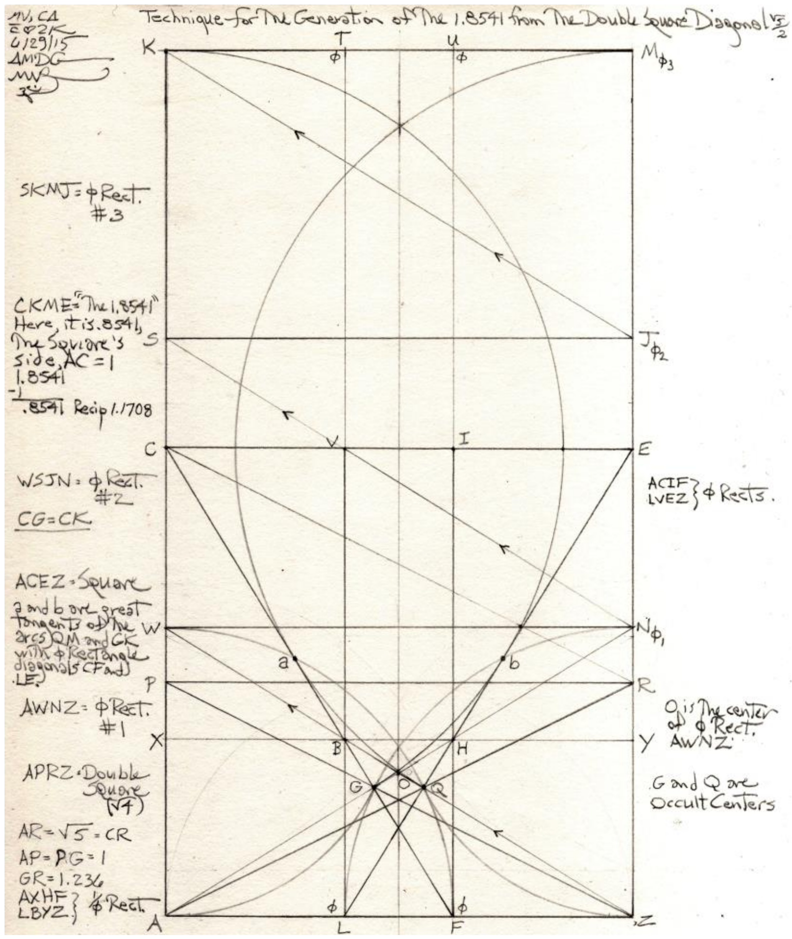


Figure 6: Generation of the 1.8541 from the Double Square and its Diagonal.

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