A Survey of Symmetry Samplers

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Abstract

In many fiber arts, such as knitting and embroidery, there is a longstanding tradition of samplers, pieces that combine patches of different designs into a single artwork. In recent years, a number of artists have produced pieces in a mathematical subgenre: the symmetry sampler, a work of art that exhibits all of the symmetry types achievable in a particular craft. This paper collects some recent examples of this emerging form and presents some of the fiber-art symmetry-group theorems these samplers demonstrate.

Samplers in the Fiber Arts

A mainstay of traditional handcrafts such as knitting, crochet, and embroidery is the sampler, a single piece comprised of patches of different designs to showcase a variety of motifs and crafting skills. In the past, these samplers were often used as instructional tools. For example, before charted lace patterns were introduced in Estonia in the 1930s, knitters would exchange patterns via knit samplers, and a woman who borrowed a friend's sampler would repay her by adding her own design to it [2].

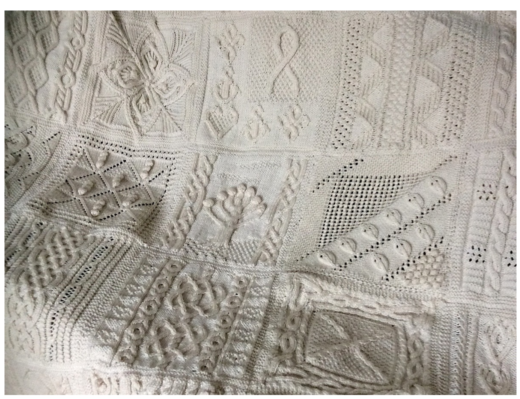

Figure 1: *The* Great American Aran Afghan*, a collaborative sampler pattern released by Knitter's Design Team. Paula Tanner knit the rendition photographed here.*

Sampler projects are still popular today as opportunities to practice a selection of different designs and techniques. One much-loved contemporary example is the *Great American Aran Afghan* (Figure 1), a blanket whose pattern squares were chosen from contest entries submitted by readers of *Knitter's Magazine* [7]. In this type of popular sampler, the designs are commonly chosen to achieve an aesthetic goal or to develop a variety of technical skills. In recent years, the mathematical fiber arts community has repurposed the traditional sampler for a more theoretical aim: to classify the symmetries of the underlying art form.

Symmetry Theorems in the Fiber Arts

A symmetry sampler is an expression of the relationship between a particular art form (for instance, color-work knitting, or blackwork embroidery) and a category of symmetry types (for instance, wallpaper groups or frieze groups). In a *complete symmetry sampler*, the sampler comprises a set of designs in which each possible symmetry type is represented exactly once.

As an example, consider *Spherical Symmetries in Temari* by Carolyn Yackel, shown in Figure 2. Temari is the Japanese art of embroidering on a sphere, and Yackel uses temari to illustrate all of the possible symmetry types of spheres [10,11]. In fact, there are infinitely many non-isomorphic spherical symmetry groups, but they fall into seven infinite families that correspond to the seven frieze groups, each of which is counted as one symmetry type, plus seven more symmetry groups related to polyhedra.

In the context of symmetries, temari is an *unconstrained* art form, because artists may start and end stitches wherever they please on the surface of the sphere. In particular, while there are practical restrictions on where and how to stitch to make a pleasing design, there are no restrictions that might preclude a spherical symmetry type. By contrast, Mary Shepherd's *Wallpapers in Cross Stitch* (Figure 3) is a complete symmetry sampler in a *constrained* art form, counted cross stitch. Counted cross stitch is a planar form executed on a square grid. As a consequence, it is not possible to realize all seventeen wallpaper groups in cross stitch: all the groups with rotations by 60° or 120° are incompatible with the underlying grid. *Wallpapers in Cross Stitch* constitutes a proof that the remaining twelve wallpaper groups are all attainable [9].

Figure 2: Spherical Symmetries in Temari*, by Carolyn Yackel, from the Bridges 2012 art exhibition. A complete symmetry sampler in an unconstrained fiber art form.*

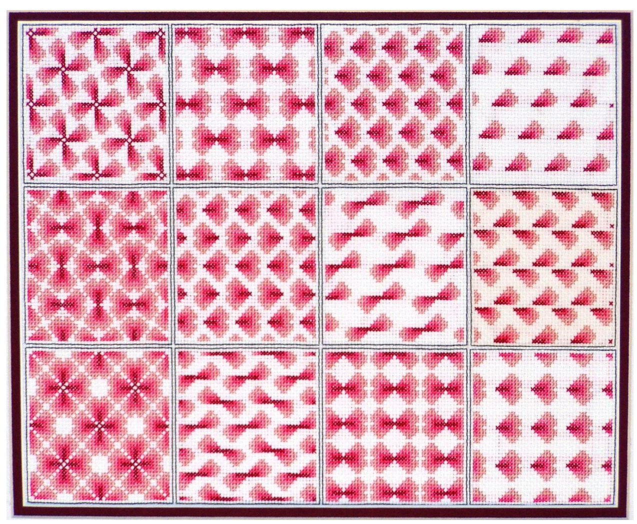

Figure 3: Wallpapers in Cross Stitch*, by Mary Shepherd, exhibited at MathFest 2007. A complete symmetry sampler in a constrained fiber art form.*

The symmetries deemed possible in a particular art form might depend on a choice of whether or not to idealize its structure. As we will see below, knitting falls into this category because of an inherent asymmetry in the knit stitch that designers traditionally ignore. A more extreme version of this actual/ideal dichotomy is apparent in *Crystallographic Bracelet Series* by Ellie Baker and the author (Figure 4). This artwork illustrates a classification of wallpaper patterns associated to bead crochet rope bracelet designs by displaying the bracelets themselves as well as the planar diagrams (the squares of printed paper) that define their patterns. The wallpaper groups enumerated here are symmetries not of the toroidal bracelets themselves but of the planar diagrams [1,4], which comprise patches of the universal covering spaces of the bracelets. Not only do most of these symmetries not apply to the bracelets themselves [4], but they also depend on an idealized representation of the beads as perfectly round and packed into a regular hexagonal grid. As the photograph in the upper right of Figure 4 shows, in practice many beads are oblate rather than spherical.

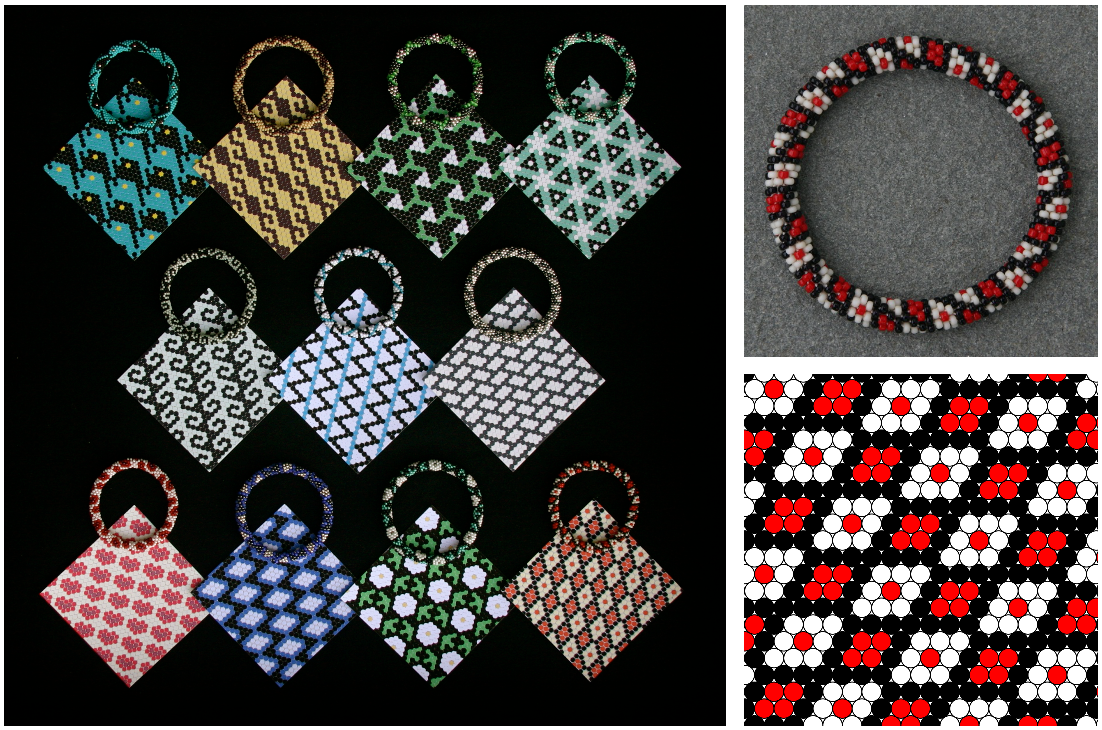

Figure 4: Crystallographic Bracelet Series*, bead crochet by Ellie Baker and Susan Goldstine, from the Joint Mathematics Meetings art exhibit in 2012. The images on the right show a close-up photo of one of the bracelets and the corresponding planar diagram. The only symmetries that apply to the bracelets themselves and not just their idealized planar designs are those of the first two bracelets in the top row.*

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One last bit of fine print: *Crystallographic Bracelet Series* is technically not a *complete* symmetry sampler. As proven in [4], exactly thirteen of the wallpaper groups are achievable in bead crochet, but only eleven of them can be produced in non-trivial ways, and the bracelet series shows those eleven symmetry types. (The remaining two groups only occur in patterns that are analogs of a monochromatic RGB display.)

A Sampling of Complete Symmetry Samplers

Except at the sewn seam that joins the beginning and end of a bracelet, bead crochet rope consists of a single type of stitch repeated throughout the flexible cylindrical tube [1,4]. With additional shaping stitches, bead crochet can be made to lie flat. *Frieze Frame*, shown in Figure 5, is a beaded disk crocheted by the author that displays all seven frieze symmetry types. The shaping comes from increases immediately before and after each pattern stripe, so that the pattern stripes themselves are effectively straight strips of beads in a hexagonal packing that have been warped into circles by the surrounding stitches. As in *Crystallographic Bracelet Series*, these symmetries treat the beads as though they are round in profile. In practice, the beads have only 180° rotational symmetry in profile and their reflection axes are oblique, so any frieze groups with reflections or glide reflections (all but the centermost and the third from the outer edge) depend on ignoring the slant of the beads.

In knitting, different variations of the art form put different levels of emphasis on the symmetries of the stitches themselves. For instance, in common styles of knitting *color work* (the generic term for knitting with more than one color of yarn at a time), pattern design glosses over the inherent asymmetry of the knit stitch. As shown in the top image in Figure 6, a close up of the double-knit scarf discussed later (see Figure 9), knit stitches have a characteristic V shape, most clearly seen in the isolated blue stitches in the centers of the brown diamonds. While this form has vertical reflection symmetry, it lacks horizontal reflection symmetry and rotational symmetry. The standard design convention is to ignore this imbalance and treat the array of stitches as a rectangular grid, so that the two rows of hearts at the top of Figure 6 are considered reflections of each other through a horizontal axis. The stitches in this fabric are noticeably shorter than they are wide—in particular, this photograph is 23 stitches wide and 23 stitches tall—making 90° rotations impossible without a significant stretch of the imagination.

In lace knitting, by contrast, the pattern is formed entirely by the shapes of the different stitches, making stitch shape an integral part of the design process. The bottom image in Figure 6 shows a close up of the lace pendant discussed later (see Figure 10). The lace design is created by *yarn overs*, which add an extra stitch and produce an open space in the knitting, and *decreases*, which both reduce stitches to compensate for the yarn overs and introduce a slant to the left or to the right. This particular design also includes raised decorative stitches called *nupps*, which create ball-like clusters of yarn and are a distinctive feature of Estonian lace.

Figure 5: Frieze Frame*, a bead crochet disk from the Bridges art exhibit in 2015.*

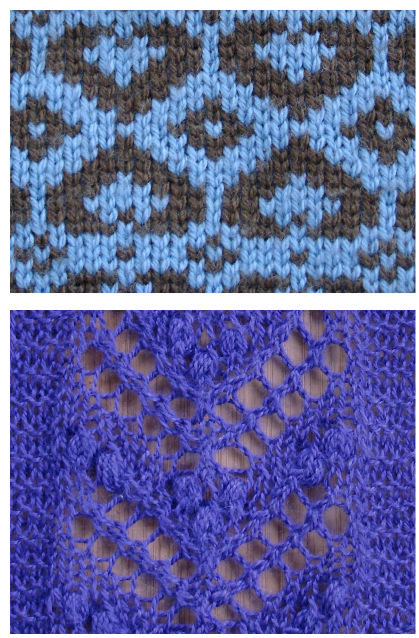

Figure 6: *Close views of stitches in knitted color work and lace.*

Fair-isle knitting, a form of color work commonly used in traditional Scandinavian sweaters, naturally lends itself to frieze patterns. (By current convention, the term *fair-isle* applies to color work in which only two colors are used at a time, and the unused color is carried in a horizontal strand across the back of the work.) *Norwegian Frieze* by Heather Ames Lewis (Figure 7) proves both a mathematical theorem and a cultural one: all seven frieze symmetries are attainable in fair-isle knitting, and all seven occur in traditional Norwegian designs [12]. (It also confirms, as does *Frieze Frame*, that frieze-based artworks are susceptible to puns.) In a similar vein, the sweater pattern *Deep Frieze* (Figure 8), in development by the author, has seven bands of fair-isle color work, one for each frieze group.

Figure 7: Norwegian Frieze*, fair-isle knitting by Heather Ames Lewis, from the Joint Mathematics Meetings art exhibit in 2016. These representatives of the seven frieze groups are all historical patterns from traditional Norwegian knitting.*

Figure 8: Deep Frieze*, a fair-isle sweater pattern in development.*

The symmetries in *Norwegian Frieze* and *Deep Frieze* treat the color pattern as a coloring of a rectangular grid, as described above. However, the V shape of the individual stitches (Figure 6) makes all symmetries that swap the top and bottom of the design (horizontal reflections, horizontal glide reflections, and 180° rotations) merely approximate. In *Norwegian Frieze*, the only symmetries that respect the stitch shape are the first and the fifth; in *Deep Frieze*, they are at the yoke and the right elbow of the sweater.

One can also consider what wallpaper groups are attainable in knitting color work. The answer is easy to deduce from Shepherd's classification of cross-stitch symmetries; since color work in knitting fits into a strictly rectangular grid, knitting can accommodate exactly the nine cross-stitch symmetries without a rotation by 90°. Figure 9 shows two renditions of this concept by the author in double knitting, a form of color work that produces matching color-swapped patterns on the two sides of the knitting. On the left is a scarf whose pattern, *Crystalline*, is published in the Deep Fall 2016 issue of *Knitty* [5], and on the right is a wall-hanging version of the same designs (with a few subtle refinements), *The Double Knitting Groups*. The wall hanging has the added feature that the symmetries become more complex as they progress down and to the right. The design in the upper left of *The Double Knitting Groups* has only translational symmetry, the three designs adjacent to it have translations and one other type of plane symmetry (going clockwise, reflections, rotations, and glide reflections), and the remaining designs have at least three types of plane symmetries.

Naturally, these designs also ignore the shape of the individual knit stitches. Only four of the symmetries respect the V shape, namely those with only translations, vertical reflections, or vertical glide reflections. In *The Double Knitting Groups*, these are the first and second in the top row, the first in the middle row, and the second in the bottom row.

Figure 9: Crystalline*, a double-knitted scarf pattern published in the Deep Fall 2016 issue of* Knitty*, and* The Double Knitting Groups*, from the Joint Mathematics Meetings art exhibition in 2017.*

Rotations, horizontal reflections, and horizontal glide reflections in knitted lace involve another layer of approximation. Here, it is not merely the individual stitch shapes that violate these symmetries, but also the placement of the stitches in the fabric as a whole. This stems from a fundamental top/bottom asymmetry in the positions of the decreases relative to the yarn overs. (The reasons for and effects of this asymmetry are too complicated to elucidate here, but will appear in a forthcoming paper on symmetries in knitting.)

Linear Lace Pendant (Figure 10), the author's most recent symmetry sampler, depicts the seven frieze groups in designs inspired by traditional Estonian lace. The three groups in the top row all contain rotations and thus are only approximate. If we refuse to ignore the shape and position of the stitches, each of these designs actually has the same frieze group as the design immediately below it in the second row. For instance, the pattern in the middle of the top row contains translations, vertical glide reflections, and the appearance of rotations and horizontal reflections, while the pattern beneath it contains only translations and vertical glide reflections. The decorative lace edging at the bottom—also a common feature of traditional Estonian lace—corresponds to a frieze group that can only be knitted at right angles to the rest of the pendant.

The current record for the most symmetries in one sampler might be held by *Simple Symmetric Symmetry Sampler*, a piece of blackwork embroidery by Wing L. Mui (Figure 11). This astounding artwork has the added feature that it depicts the smallest simple perfect squared square, i.e., the smallest square that can be subdivided into smaller squares no two of which are the same size [3,6]. Within the subsquares of Mui's square are all of the seven frieze groups (all in the largest square in the upper left), twelve wallpaper groups, and six rosette groups compatible with the square grid of blackwork embroidery, for a total of twenty-five symmetry types.

Figure 10: Linear Lace Pendant*, in the Bridges 2017 General Exhibit, a knitted pendant of the seven frieze groups in patterns inspired by Estonian lace (as in [2]).*

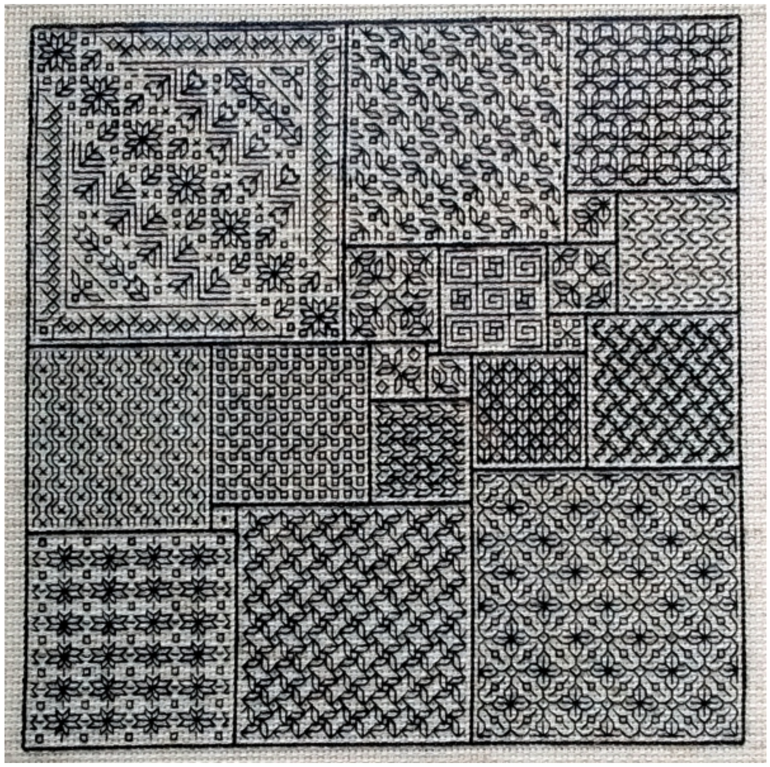

Figure 11: Simple Symmetric Symmetry Sampler*, blackwork embroidery by Wing Mui, from the Joint Mathematics Meetings art exhibit in 2016. This perfect simple squared square features all of the frieze, wallpaper, and rosette symmetries possible in blackwork embroidery.*

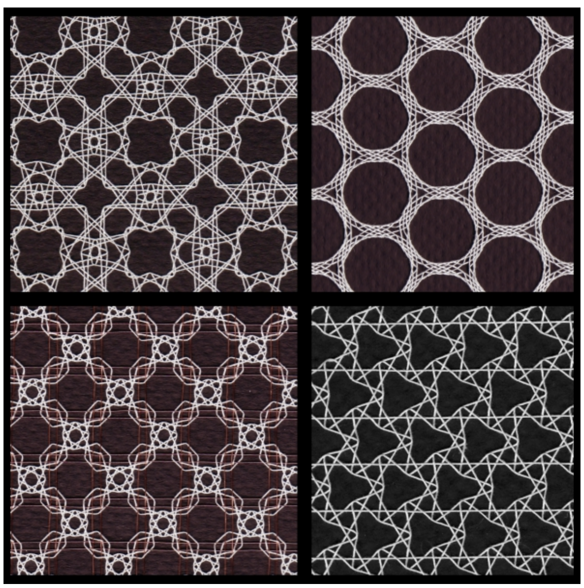

Figure 12: Speculations*, bobbin lace by Veronika Irvine and Lenka Suchanek, from the Joint Mathematics Meetings art exhibit in 2016. This shows three of the seventeen wallpaper groups (the two designs on the left have the same symmetry group), but all seventeen are possible in bobbin lace.*

Future Directions

Given the current level of activity in mathematical fiber arts, it seems inevitable that artists will produce more complete symmetry samplers, and hence more symmetry theorems in the fiber arts. In fact, one art form whose symmetries were recently classified is bobbin lace, the subject of the remarkable work of Veronika Irvine and Frank Ruskey [8]. Figure 12 shows *Speculations*, a piece by Irvine and Lenka Suchanek, with three different symmetry groups in bobbin lace. Irvine and Ruskey have proven that all seventeen wallpaper groups are possible, suggesting that a complete symmetry sampler in bobbin lace is on the horizon. There are also other forms of knitting to investigate: in conversation, Heather Ames Lewis has described a current investigation into cabling patterns. And there are so many further fiber arts to explore. With all the mathematically inclined artists who are currently knitting, crocheting, embroidering, weaving, tatting, sewing, and so forth, the future of symmetry samplers is full of promise.

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