The Size and Shape of Utopia

Andrew Simoson
King University
Bristol, Tenessee, 37620 USA
`ajsimoso@king.edu`

Abstract

In celebration of the fifth centennial of Sir Thomas More's *Utopia* of 1516, we offer a solution to a geographical/mathematical puzzle posed in the text, namely, the possible size and shape of the island. Translating More's five verbal clues into mathematics yields five equations with five unknowns, and a solution to the puzzle, which we contrast with various maps made by cartographers/artists over the years.

This year marks the fifth centennial of the publication of Sir Thomas More's *Utopia* in 1516. As a partial celebration of this occasion, we focus on More's curious mathematical puzzle about the size and shape of Utopia Island from the opening lines of Book II of his fantasy about what might make a more perfect society. Here are a few pertinent lines translated from the original Latin by Gilbert Burnet in 1684 [5], as well as the woodcut of Utopia Island from the first edition of 1516, Figure 1:
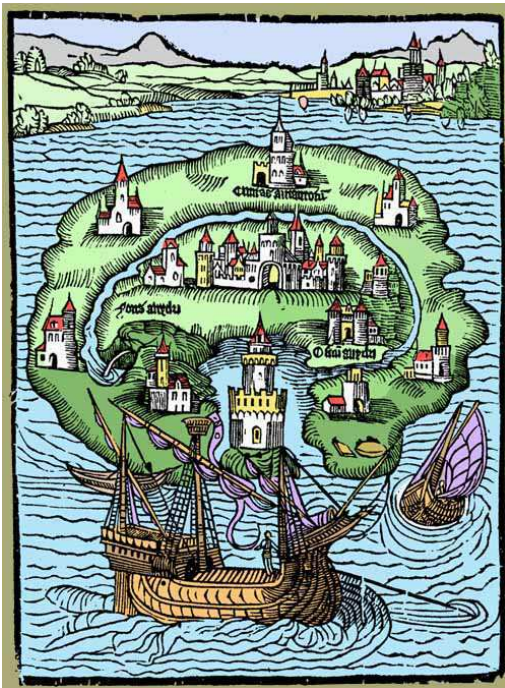

Figure 1: *Utopia, 1516, courtesy of Yale's Beinecke Rare Book Library.*

"The island of Utopia is in the middle two hundred miles broad, and holds almost at the same breadth over a great part of it, but it grows narrower towards both ends. Its figure is not unlike a crescent. Between its horns the sea comes in eleven miles broad, and spreads itself into a great bay, which is environed with land to the compass of about five hundred miles. The entry into the bay, in the middle of it, [has] one single rock; and on the top of it is a tower, in which a garrison is kept. [Long ago, the first king of Utopia], Utopus, ordered a deep channel to be dug, fifteen miles long [to separate Utopia from the mainland].

"There are fifty-four cities in the island, and they are all contrived as near in the same manner as the ground on which they stand will allow. The nearest lie at least twenty-four miles' distance from one another. Amaurot is the chief town of the island, being situated near the centre of it, so that it is the most convenient place for their assemblies. The jurisdiction of every city extends at least twenty miles.

"Amourot lies upon the side of a hill. Its figure is almost square, for from the one side of it, which shoots up almost to the top of the hill, it runs down, in a descent for two miles, to the river Anider. The Anider rises about eighty miles above Amaurot, in a small spring at first. As it runs by Amaurot it is grown half a mile broad; but, it still grows larger and larger, till, after sixty miles' course below it, it is lost in the ocean."

Distilling More's description of Utopia's size and shape from these three *Utopia* paragraphs, we refine the passage as five clues while making some assumptions about the orientation of the cross-sections and the direction of the River Anider.

Clue 1: Utopia is shaped as a crescent, the horns of which bound a large harbor on its eastern end.

Clue 2: If we loosely define the *midline* of the island as the perpendicular bisector of the crescent's two tips, then Utopia's cross-sections parallel to the midline are all about 200 miles, except near the extremes where it collapses to 0.

Clue 3: This harbor is circular with a mouth of 11 miles so as to make a perimeter of about 500 miles.

Clue 4: Utopia consists of 54 city-states, each separated from the nearest neighbor by 24 miles. Each city-state is square-like with side lengths of at least 20 miles.

Clue 5: The capital city, located in the center of the island, lies about 60 miles from the harbor and 140 miles from the opposite coast.

Figure 2: *Utopia, 1595, by Abraham Ortelius, courtesy of Thomas Fisher Rare Book Library, University of Toronto.*

Over the centuries, and usually near a centennial of the 1516 appearance of *Utopia*, artists have tried to render Utopia Island following More's five clues. Yet for various reasons, probably chief among them artistic license, none of the maps satisfy all five. Some artists have gone so far as to comment on their failures. For example, Goodey [2] interprets Utopia's 200 mile cross-sections as being perpendicular to the midline of the island through the bay; in this case the horns of the island's crescents become rather stubby; and he ultimately concludes that More's island clues cannot be taken literally. McClung [4] more or less dismisses "mapping efforts [from More's clues] as futile." One reason for their gloomy conclusions is that they interpret Utopia island as having a perimeter of 500 miles and a diameter of 200.

Meanwhile, we critique four classic maps of Utopia from 1516, 1595, 1704, and 1717, Figures 1, 2, 4, and 5, respectively.

Map of 1516 (from the first edition), Figure 1. This map is the most charming rendition of Utopia of the four. Its shape is roughly circular. Unlike the other maps, the River Anider follows a semi-circular route through the island, beginning as a spring in the bottom left horn, flowing about 80 miles to the capital city, and then 60 miles more to empty into the central bay. Unfortunately, when straightened, this river of 140 miles is more than 200 miles, violating Clue 2. The horns of the island are somewhat pudgy, reminiscent of

the dilemma expressed earlier by Goodey. Furthermore, only ten cities are illustrated, yet the artist has given us a flavor of Clue 4; insertion of any more cities into the woodcut would merely add unnecessary clutter.
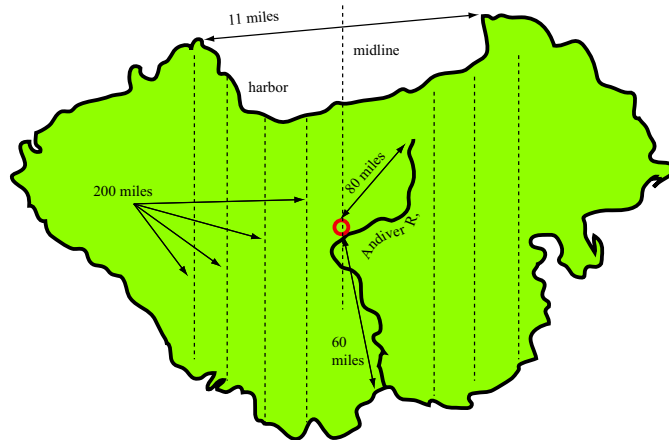

Figure 3 : *Outline of Ortelius's Utopia, cross-sections parallel to the midline.*

Map of 1595, Figure 2. This map is the most elegant of the four. It was drawn by Abraham Ortelius, a Flemish cartographer who is credited with producing the world's first modern atlas, *The Theatre of the World*; as a sidenote, he was the first to propose that the world's land masses were once an *urcontinent* before drifting over the eons to their present locations. Although Figure 2 is too small to see the detail, Ortelius has labeled the fiftyfour cities, fully satisfying Clue 5. As indicated in the outline of his map, Figure 3, his 200 mile cross-sections are indeed parallel to the midline of the island through the bay. His River Andiver starts just south of the bay, flows for 80 miles to the capital and empties 60 miles downriver on the southern coast. However, if we straighten the river, its length is at least 200 miles. Furthermore, with respect to the 200 mile cross-sections, the tips of the horns to the bay are far more than the specified 11 miles, violating Clue 3.

Incidentally, More's Utopia Island description clues are thin camouflage for Henry VIII's England. That is, the 1577 *Description of England* compiled by cleric William Harrison describes England and Wales as comprised of 53 counties and London, making 54 city-states, [3]. Furthermore, Utopia's fifteen mile channel is suggestive of the far greater English Channel. Yet to add more similarities, Sir Thomas exercised discretion lest he lose his head before his time.
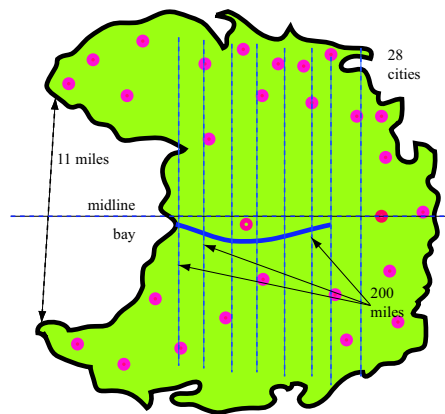
a. Courtesy of Yale's Beinecke Rare Book Library.   *b. Outline, cross-sections ⊥ midline.*
Figure 4 : *Utopia Island, 1704 edition.*

Map of 1704, Figure 4. This map displays only 28 cities, although the artist could easily have added more.

His 200 mile cross-sections are perpendicular to the midline as indicated by the dotted vertical lines in Figure 4b. His length of the River Andiver is consistent with these vertical cross-section lengths. However, the horns of the crescent are much further than eleven miles apart. But of course, the artist wished to portray the tower atop the rock in the harbor mouth.

Map of 1717, Figure 5. This map is the most dramatic of the four, and comes from a French edition, lavishly illustrated. The wondrous detail of ships and hills and buildings in the figure remind one of the video games where players create virtual settlements, which grow to villages, which grow to cities, and then to megalopolises. The crescent bay is neatly rendered. But incorporation of the other clues are nonexistent.


Figure 5 : *Utopia, from a 1717 edition, courtesy of the Peace Palace Library, The Hague.*

Since we could stack 200-mile-cross-section-lengths indefinitely and so attain a Utopia Island map of arbitrarily large area satisfying all five clues, our strategy in solving More's geography puzzle is to fashion an island of minimal area satisfying all five clues. To achieve this end, we formalize the five clues:

Feature 1: The outer coast of Utopia will be an ellipse parameterized by $\mathcal{O} = (a \cos t, \, b \sin t)$.

Feature 2: The inner harbor coast will be an ellipse $\mathcal{H} = (c \cos t + 100, \, d \sin t)$ where $c > d$ and d is the radius of the osculating circle at the right-hand side of $\mathcal{O}$. By *osculating circle* we mean the inscribed circle of largest radius that is tangent to the ellipse at its extreme right-hand point.

Feature 3: To ensure that the inner harbor has a perimeter of 500 miles, we exchange the osculating circle for an ellipse $\mathcal{H}$ of semi-axial lengths c and d where c slightly exceeds d.

Feature 4: Each city-state has a central downtown disk region of radius one mile so that the minimum distance between the centers of any two neighboring city-states is at least 26 miles, (while the minimum distance between neighboring downtowns remains at 24 miles). Furthermore, the capital city is 60 miles from the east coast and 140 miles from the west coast.

Feature 5: The capital city will be located at the *centroid* of this island.

The *radius of curvature $r(t)$* of ellipse $\mathcal{O}$ at parameter t is

$$r(t) = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}{ab},$$
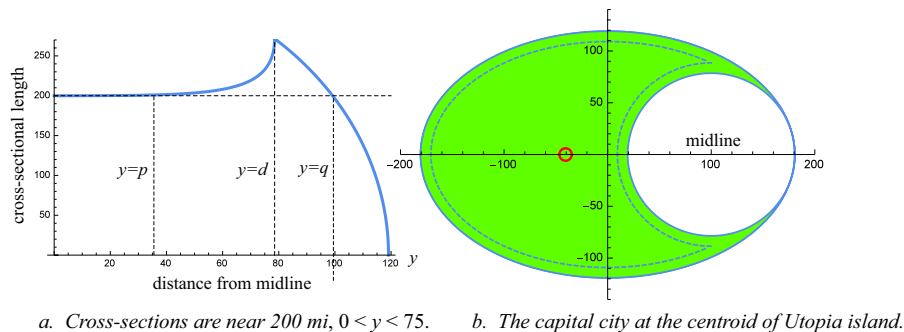
a. *Cross-sections are near 200 mi, $0 < y < 75$.*   b. *The capital city at the centroid of Utopia island.*

Figure 6 : *A possible size and shape for Utopia.*

as found, for example, in [1, pp. 28–30]. Thus at $t = 0$, Features 1 and 2 imply that the radius d of the osculating circle to $\mathcal{O}$ must be

$$d = r(0) = \frac{b^2}{a}, \tag{1}$$

which we take as the semi-axial height of the elliptical inner harbor $\mathcal{H}$. Since the major axis of the island has length $2a$ as well as $200 + 2c$ then Features 1 and 2 again imply

$$a = 100 + c. \tag{2}$$

By Feature 3, the perimeter of the harbor is 500 miles,

$$\int_0^{2\pi} \sqrt{c^2 \sin^2 t + d^2 \cos^2 t} \, dt = 500. \tag{3}$$

By symmetry, Utopia's centroid must be located at $(-s, 0)$ for some nonnegative number s. Since the distance from the capital city to the harbor is 60 miles and the centers of $\mathcal{O}$ and $\mathcal{H}$ are at $(0, 0)$ and $(100, 0)$ respectively, then Features 4 and 5 give

$$s + (100 - c) = 60. \tag{4}$$

Think of the inner harbor $\mathcal{H}$ as having negative area $(-\pi cd)$ centered at $(100, 0)$ and the outer ellipse $\mathcal{O}$ as having positive area πab centered at $(0, 0)$. Therefore Feature 5 becomes

$$s(\pi ab) + (100 + s)(-\pi cd) = 0. \tag{5}$$

Numerically solving the system of equations (1) through (5) gives

$$a \approx 180.62, \, b \approx 119.09, \, c \approx 80.62, \, d \approx 78.53, \, s \approx 40.62.$$

With these parameter values, Figure 6a shows that Utopia's cross-sectional length is near 200 miles for distance y from Utopia's midline, $0 \leq y \leq 70$, and plunges to 0 as y goes from about 95 to $b \approx 119.09$. Since each city-state is at least 20 miles along each side, each city must be centered at least 10 miles inland from the coast, a boundary marked by the dotted curves along Utopia's shoreline in Figure 6b. The capital city at this region's centroid is marked as a small open circle. For this island size and shape, an approximate greatest lower bound for Utopia's area is $\pi(ab - cd) \approx 47\,690$ mi^2, uncannily near the area of modern day England (excluding Wales), $50\,346$ mi^2.

Before placing the remaining 53 city-states as dots within the island, which we presently do, stacking them as two dimensional cannonballs or oranges, we pause to insert twin reservoirs in Utopia whose centroids lie along and are symmetric to the vertical line through the capital city so that the cross-sectional spike in
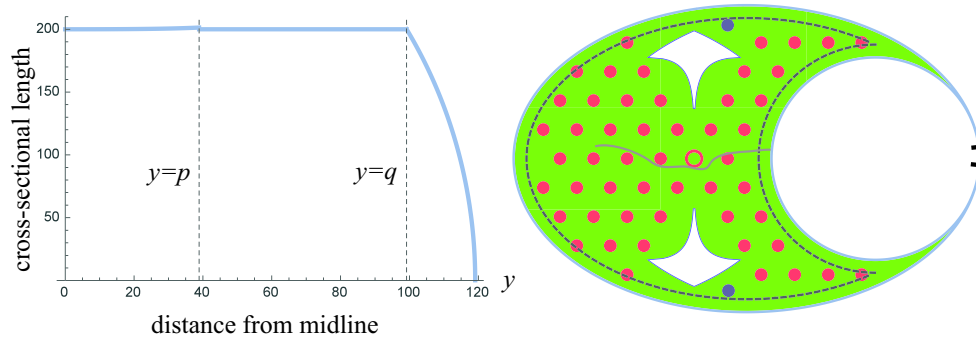
a. Cross-sections are near 200 mi, $0 < y < q$.   *b. All 54 city-states and two reservoirs.*
Figure 7 : *A better Utopia size and shape, with twin reservoirs.*

Figure 6a is flattened to a constant cross-section of 200 miles, on the interval $p \leq y \leq q$, where the cross-sectional widths of the island of Figure 6 at $p \approx 35.55$ miles and $q \approx 99.18$ miles are 201 and 200, respectively. After this engineering feat, the capital city yet remains at the centroid. However, since each reservoir has approximate area of 1297 mi^2 by Cavalieri's principle, then our revised infimum for Utopia's area is $45\,096$ mi^2. Figure 7a shows that the cross-section width of this modified island is indeed near 200 miles, $0 \leq y \leq q$, and tails to 0 afterwards. Figure 7b shows a distribution of 54 city-states as a series of dots.

Finally in Figure 8, to reduce the island's area further, we add fjords from the reservoirs to the sea and allow the shoreline to meander in a mild fractal manner, much like the shorelines of our four classic Utopian maps, thereby adding a natural touch of elegance.
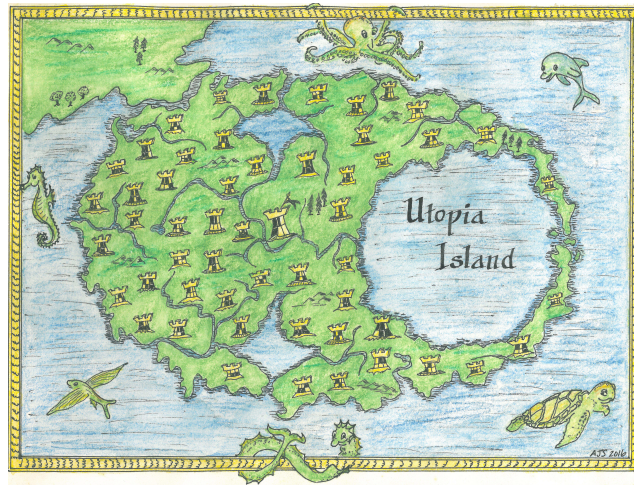

Figure 8 : *Utopia map, in celebration of Utopia's fifth centennial.*

References

[1] J. W. Bruce and P. J. Giblin, *Curves and Singularities*, Cambridge University Press, 1984.

[2] B. R. Goodey, Mapping *Utopia*: a comment on the geography of Sir Thomas More, *Geographical Review* 60 (1970) 15–30.

[3] William Harrison, *The Description of England: The classic contemporary account of Tudor social life*, Georges Edelen, ed., Courier Corporation Press, Ithaca, 1968.

[4] W. A. McClung, Designing Utopia, *Moreana* 31: 118–119 (June 1994) 9–28.

[5] Sir Thomas More, *Utopia*, Henry Morley, ed., Cassel & Company Press, London, 1901.