Origami as a Tool for Exploring Properties of Platonic Solids

Natalija Budinski*
Primary and secondary school “Petro Kuzmjak”,
25233 Ruski Krstur,
Serbia
nbudinski@yahoo.com

Abstract

This workshop will elaborate how to introduce students (upper elementary and high school students, age 14-18) to Platonic solids and their properties through origami activities. Some of the important mathematical concepts related to these well known geometrical solids will be shown to the participants. The participants will be given instructions how to create origami Platonic solids, and materials related to their mathematical properties will be provided.

Introduction

It is well known that origami is an ancient Japanese technique of folding paper that has proven to be a very important mathematical concept. When the origami axioms as mathematical principles of folding paper were set, see [1], many of the geometrical constructions that were impossible to make with the straight edge and compass became solvable. The researchers noticed many excellent educational properties of origami, see [2], [11], [4], [6], [7]. It is well known that the construction of polyhedra is a path of developing spatial relations and geometry understanding. The benefit lies in the process of making a model, not just holding a pre-made one. By assembling models by their own hands, the students develop a sense for properties of the objects, see [9]. In geometry lessons, this can be achieved by the use of modular origami, where the students are given a chance to explore the features of polyhedra by assembling them from parts made by origami techniques.

Platonic Solids Made by Origami

The workshop will give practical guidelines how to introduce students to Platonic solids as a specific class of regular solids. The introduction is based on the fact that the aid of visual imagination can help students discover facts without entering into formal definitions and concepts, see [8]. It will be shown that activities involved with origami help the students open a mathematical discourse and talk about geometry and spatial relation through the manipulation of paper or model modifications.

A Brief Overview of Platonic Solids. Firstly, the workshop participants will be introduced to regular convex solids known as Platonic solids. They are called Platonic after the ancient philosopher Plato, who associated them with classical elements such as earth, air, water and fire. Besides Plato, Platonic solids have been the inspiration for many ancient and contemporary artists, such as Leonardo, Barberi, Durer, Emmer, Escher, Hohl. There are only five Platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. The tetrahedron is the simplest Platonic solid, consisting of four equilateral triangle faces. The cube is a Platonic solid with six square faces, also called hexahedron. The octahedron has eight equilateral triangle faces, the dodecahedron has twelve faces represented by regular pentagons, while the icosahedron has twenty faces represented by equilateral triangle faces. As well as in art and mathematics, Platonic solids can be found in nature. This activity of introducing participants to Platonic solids lasts approximately 10 minutes.

Assembling Platonic Solids. Secondly, after connecting with the divinity of Platonic solids through history, the solids will become alive through paper folding. The workshop participants will be given instructions on how to fold units and assemble them into Platonic solids. As certain skills in paper folding are required, parts or models made beforehand will be provided to make the process more efficient. When applying origami in school, particularly when making sophisticated models, the students can do some paper folding as homework. In order to make Platonic solids the origami techniques proposed by Glassner, see [5] will be used. The tetrahedron is built from the pieces in rectangular form with sides ratio 2:1. The instructions of folding tetrahedron units are shown in Figure 1. The tetrahedron is made from six units and it is represented in the Figure 2.

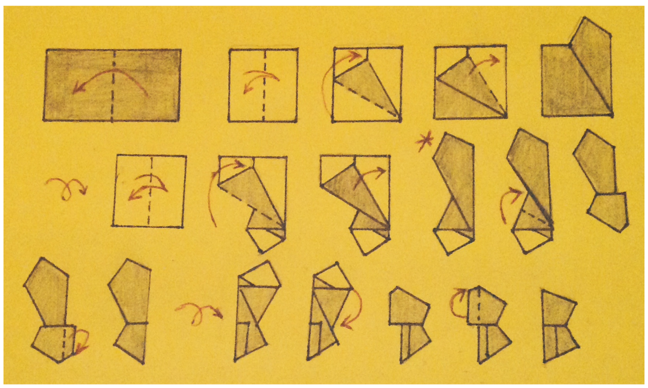

Figure 1: *Instructions for folding tetrahedron units*

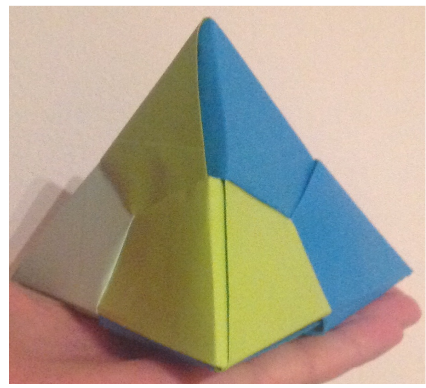

Figure 2: *Tetrahedron made by origami*

The cube is built from the square form pieces, which should be folded as in Figure 3. For assembling the cube 12 units should be made. Figure 4 represents cube made by origami.

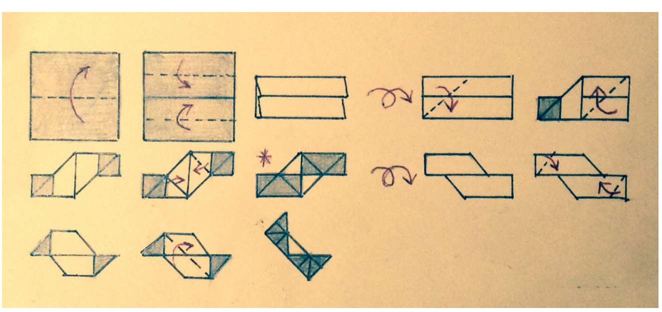

Figure 3: *Instructions for folding cube units*

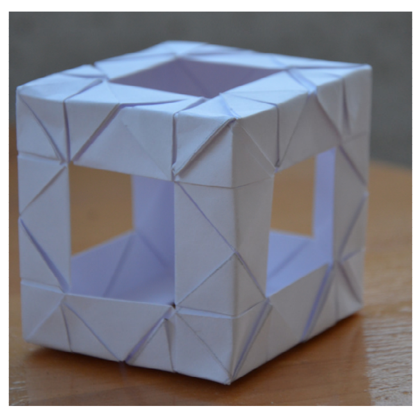

Figure 4: *Cube made by origami*

The octahedron can be assembled from 12 units presented in Figure 1. The octahedron made by origami is represented in the Figure 5.

Figure 5: *Octahedron made by origami*

For assembling the dodecahedron twenty triangular vertex units should be made as in Figure 6. Figure 7 represents dodecahedron made by origami.

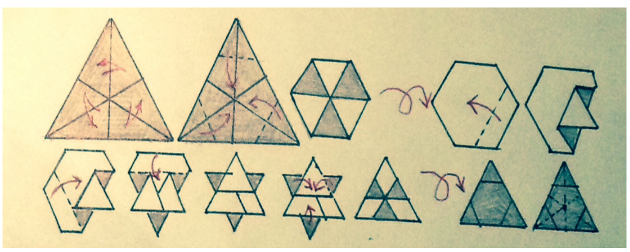

Figure 6: *Instructions for folding dodecahedron units*

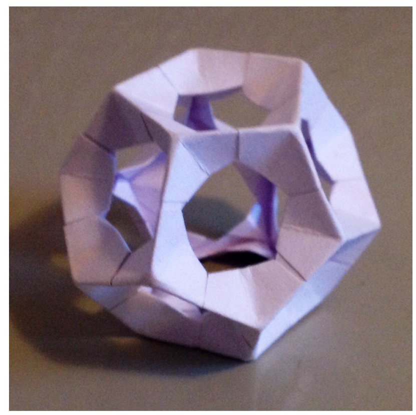

Figure 7: *Dodecahedron made by origami*

Being the most complicated, the icosahedron needs ten double face equilateral triangles, as shown in Figure 8. The icosahedron made by origami is represented in the Figure 9.

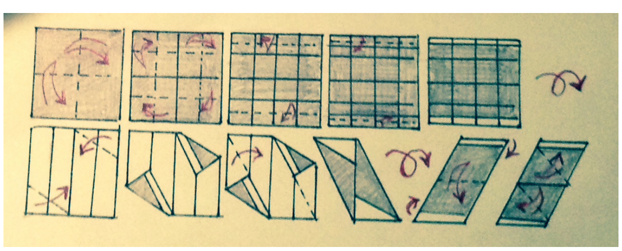

Figure 8: *Instructions for folding icosahedron units*

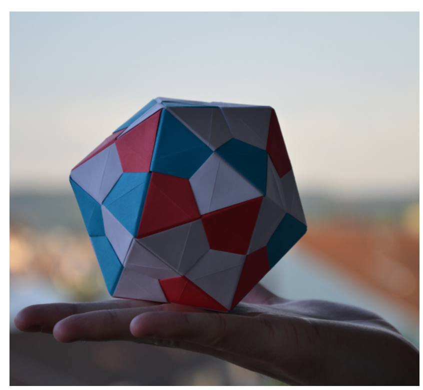

Figure 9: *Icosahedron made by origami*

The activity of assembling Platonic solids by origami lasts approximately 50 minutes. It can be done by groups, where each group would assemble one solid.

Mathematical Properties of Platonic Solids

In this part of the workshop, models made by origami would be used for the explanation of mathematical properties of Platonic solids, and polyhedra in general. Firstly, participants would be introduced to basic mathematical properties of Platonic solids, such as faces (F), edges (E) and vertices (V). Faces are polygons of which the polyhedron is made. The faces meet along their edges, and the edges meet at the vertices which are the points. Also, the participants would be introduced to the definition of regular polyhedrons. The definition says that polyhedra whose faces are equal are regular. Furthermore, the participants should be introduced to the property of convexity. Regular polyhedra are convex, which means that the line segment obtained by connecting any two points on the polyhedron surface is on the surface of the polyhedron. Also in the case of convex polyhedra there are not any self intersections of polyhedron surface. The participants will be able to check that each vertex of the Platonic solid has the same number of meeting faces. Those definitions, even though seemingly obvious, are necessary in the introductory part.

Secondly, the participants might be wondering why are there only five Platonic solids. The proof, that would be visualized by the origami made model, is based on the fact that there are at least three faces at each solid vertex. In the process of subtracting internal angles that meet at the vertex, the sum must be less than 360°. It is because at 360° the shape will be flattened. The Platonic solids consist of identical regular polygons, such as: triangles with 60° internal angles, squares with 90° internal angles, or pentagons with 108° internal angles. Hexagons cannot be the faces of Platonic solids because their internal angles are 120°, and meeting three hexagons in the vertex will flatten the shape. So, when three regular triangles meet at the vertex, they constitute the 180° angle of the tetrahedron. Four regular triangles met in the vertex constitute the 240° angle of the icosahedron. Three squares met in the vertex constitute the 270° angle of the cube, and three pentagons met in the vertex constitute the 324° angle of the dodecahedron. Anything else would constitute an angle of 360° or more, which is impossible. For example, four regular pentagons would constitute a 432° angle.

Thirdly, the participants will be introduced to numerous mathematical facts and formulae regarding Platonic solids. Some of them are in Table 1.

Polyhedron	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Dihedral angles	$\arccos\left(\frac{1}{3}\right)$	$\frac{\pi}{2}$	$\pi-\arccos\left(\frac{1}{3}\right)$	$\pi-\arccos(2)$	$\pi-\arccos\left(\frac{\sqrt{5}}{3}\right)$
Area	$\sqrt{3}a^2$	$6a^2$	$2\sqrt{3}a^2$	$3\sqrt{25+10\sqrt{5}}a^2$	$5\sqrt{3}a^2$
Volume	$\frac{a^3}{6\sqrt{2}}$	a^3	$\frac{\sqrt{2}}{3}a^3$	$\frac{15+7\sqrt{5}}{4}a^3$	$\frac{5(3+\sqrt{5})}{12}a^3$
Circumradius	$\frac{\sqrt{6}}{4}a$	$\frac{\sqrt{3}}{2}a$	$\frac{a}{2}\sqrt{2}$	$\frac{a\sqrt{3}}{4}\left(1+\sqrt{5}\right)$	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$
Inradius	$\frac{a}{\sqrt{24}}$	$\frac{a}{2}$	$\frac{a}{6}\sqrt{6}$	$\frac{a}{2}\sqrt{\frac{15}{2}+\frac{11}{10}\sqrt{5}}$	$\frac{\sqrt{3}(3+\sqrt{5})}{12}a$

Table 1: *Mathematical formulae related to Platonic solids with edge length a*

For example, there are values of dihedral angles, which are interior angles formed by two face planes, also there are formulas for area, volume, circumradius and inradius. Circumradius is the radius of a sphere that passes through the vertices of a polyhedron called circumscribed sphere. Inradius is the radius of a sphere which is tangent to the faces of a polyhedron at the centre of the face.

At the end of the workshop, the participants would be guided to discover Euler's theorem for convex polyhedrons. This formula was established in 18th century by Euler. In order to be introduced to Euler's theorem, the participants would be asked to fill in the following Table 2. The origami made models can be used as a visual remedy in the process of filling Table 2.

Polyhedron	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Vertices (V)					
Edges (E)					
Faces (F)					
V-E+F					

Table 2: *Task for the workshop participants*

If the participants correctly fill in Table 2, they will obtain the number two in every field of the last row. The expression derived from Table 1 is called Euler's theorem. This theorem holds for convex and some non-convex polyhedrons. The relation between the number of polyhedron's vertices (V), faces (F) and edges (E) can be described as V-E+F=2. The activity of discovering mathematical properties lasts approximately 30 minutes.

Conclusion

It is evident that the idea of using origami in mathematical education has sprouted in the education circles. Now, it is needed to grow and make it a part of the common educational practice. The proposed workshop gives us a chance to inform and educate attendees on how to implement origami in geometry lessons, regardless of being teachers or just origami, art or mathematics fans. Platonic solids have been studied throughout centuries due to their mathematical and aesthetic beauty. Origami provides techniques with which models of Platonic solids can be made and gives an opportunity to explore their properties empowered with visual representations. One of the leading experts in computational origami Robert Lang, see [10], said that as much as it sounds incredible, origami might save lives one day. The proposed workshop will show how it can be also helpful with the improvement of teaching students geometry.

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