Dual Models: One Shape to Make Them All

Mircea Draghicescu
ITSPHUN LLC
mircea@itsphun.com

Abstract

We show how a potentially infinite number of 3D decorative objects can be built by connecting copies of a single geometric element made of a flexible material. Each object is an approximate model of a compound of some polyhedron and its dual. We present the underlying geometry and show a few such constructs. The geometric element used in the constructions can be made from a variety of materials and can have many shapes that give different artistic effects. In the workshop, we will build a few models that the participants can take home.

Introduction

Wireframe models of polyhedra can be built by connecting linear elements that represent polyhedral edges. We describe here a different modeling method where each atomic construction element represents not an edge, but a pair of edges, one belonging to some polyhedron P and the other one to P', the dual of P; the resulting object is a model of the compound of P and P'. Using the same number of construction elements as the number of edges of P and P', the model of the compound exhibits a richer structure and, due to the additional connections, is stronger than the models of P and P' alone. The model has at least the symmetry of P and P', and can have a higher symmetry if P is self-dual.

The construction process challenges the maker to think simultaneously of both P and P' and can be used as a teaching aid in the study of polyhedra and duality. These topics will be discussed in the workshop, where we will also build a few such models.

Finally, we show that the constructs presented here generalize some well known decorative lamps.

Geometry

The models described here preserve the connectivity relations between the vertices, edges, and faces of a polyhedron. These relations, which are not affected by polyhedra distortions, are captured by the associated *polyhedron graph*; it would thus be more appropriate to say that we model polyhedra graphs. For simplicity, we will still use the word "polyhedron" and the names of known polyhedra to refer to the associated graphs.

The (topological) *dual* P' of a polyhedron P has, by definition, a vertex for each face of P and an edge between the vertices that correspond to faces of P which share an edge. The mapping that takes an edge e of P to its *dual edge* e' of P' that connects the vertices corresponding to the faces separated by e is one-to-one; a polyhedron and its dual have the same number of edges. The dual of P' is P and the the dual edge of e' is e. P is *self-dual* if P and P' are the same polyhedron; examples include any *pyramid* (in particular the *tetrahedron*) and any *elongated pyramid*, the object obtained by adjoining a pyramid to the base of a prism.

A *dual compound* is the figure obtained by placing dual polyhedra P, P' so that the dual edges intersect each other. We will model here such compounds using construction elements that model dual pairs of edges e, e'. Each construction element has 4 connection points that correspond to the endpoints of e and e'. In the resulting model, the connection points that correspond to the endpoints of e are attached to connection *nodes* in the model that correspond to two vertices of P (or, same thing, two faces of P'), while the other

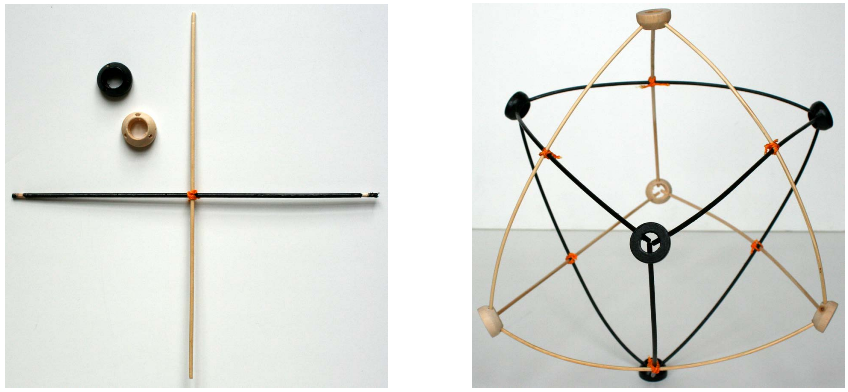

Figure 1: *Simple wood cross-shaped construction element, tetrahedron + tetrahedron (Stella octangula) model*

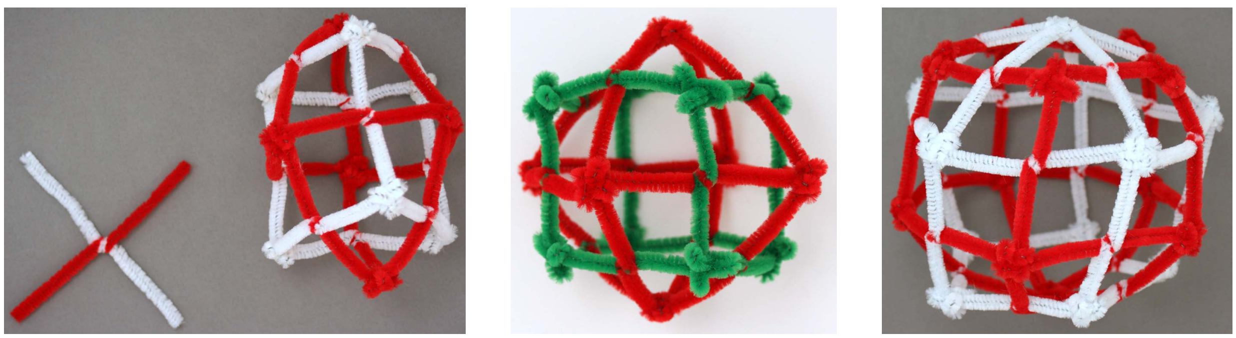

Figure 2: *Pipe cleaner models: (a) triangular prism (white) + triangular bipyramid (red), 9 elements, (b) cube (green) + octahedron (red), 12 elements, (c) elongated pentagonal pyramid (self dual), 20 elements*

two connection points are attached to nodes that correspond to two vertices of P' (or two faces of P). The nodes of the complete model correspond, depending on your point of view, to vertices of both P and P', or to their faces, or to the vertices and faces of P (or P'). If the polyhedron P satisfies Euler's formula (i.e., if P is a topological sphere) then the number of nodes is 2 plus the number of construction elements.

Models

Figure 1 shows a simple cross-shaped construction element made of wood where the two modeled edges are clearly visible. Using 6 such elements and 8 wooden rings with holes as nodes, we built a (rounded) model of the compound of two tetrahedra, or *Stella octangula*. Figure 2 shows a few models made of 10 cm sections of "pipe cleaners". Two such pieces are twisted together to form a cross-shaped construction element; their ends are bent into hooks that are used to connect them in a circular manner around each node.

The flexibility of the construction elements allows the modeling, as described above, of two edges, that, in general, have different lengths and/or do not lie in the same plane, no matter how we distort the polyhedra P and P'. Stella octangula is unique in this respect since the dual edges of the two tetrahedra, which have the same length, also lie in the same plane and are perpendicular to each other. The model in Figure 1 could have been thus built without any bending.

The paper construction element shown in Figure 3 is about 12 cm in diameter; its shape and connection method was inspired by [1], [2], and the IQlight® system described below. The natural elasticity of the

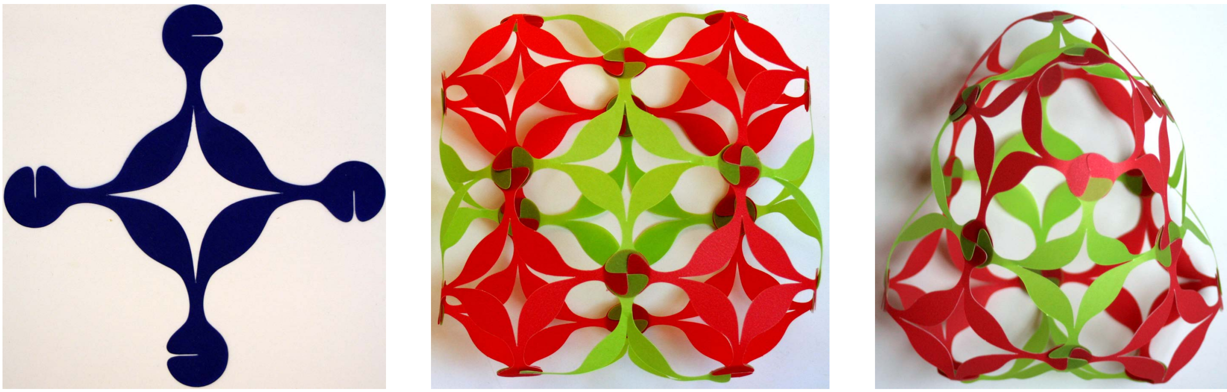

Figure 3 : *Paper construction element, bifastigium, gyrobifastigium (14 elements)*

Figure 4 : *Some models with 16 elements: elongated square pyramid (self-dual), square antiprism + tetragonal trapezohedron, and a self-dual variant with 4 triangular and 5 quadrilateral faces, half square antiprism, half tetragonal trapezohedron*

Figure 5 : *Cuboctahedron + rhombic dodecahedron (24 elements), icosahedron + dodecahedron (30 elements)*

Figure 6: *Truncated octahedron + tetrakis hexahedron (36 elements), rhombicuboctahedron + deltoidal icositetrahedron (48 elements)*

paper creates the outward directed tension that gives the models their shape. The two modeled edges, which are represented implicitly by the diagonals of the construction element, cannot be distinguished by color anymore. This makes it harder to recognize the two polyhedra in the compound model - the colors have now only an artistic purpose.

Figure 3 also shows two models with 14 elements each: *bifastigium*, the self-dual object obtained by adjoining two uniform triangular prisms on a common square face[1] and *gyrobifastigium* where the two joined triangular prisms are rotated by 90 degrees with respect to each other. The dual of the gyrobifastigium is the object obtained by dividing two opposite faces of a cube into triangles with two non-parallel diagonals (if the diagonals are parallel we obtain a bifastigium). Figures 4, 5, 6, and 7 show a few more paper models. Figure 7 shows the largest model (about 40 cm in diameter) we built so far.

In principle, we can model in this way the compound of any polyhedron P with its dual P'. Indeed, we can start by building a model of P using the construction elements as simple edges (connecting them just at two diagonally opposed connection points). We then join together the connection points corresponding to each face of P to obtain the model of P' and thus a model of the compound. In practice, the shape, material, and connection method of the construction elements, impose physical constraints on what models can be built (limiting, for example, the maximum number of sides of a polygonal face of P and P').

By bending the paper pieces we can push certain nodes either in or out of the model - Figures 5, 6, and 9 show both cases. Figure 8 shows two constructs which combine two different models with the same symmetry. Certain nodes of the smaller, inner model are pushed out and are interweaved with the corresponding nodes of the larger, outer model that are pushed in.

Related system: IQlight®

In 1973, the Danish designer Holger Strøm invented a system of interlocking geometric forms that can be used to assemble decorative objects (usually lamp shades) in an infinite number of shapes. The plastic pieces are easy to reproduce and are available for purchase from many sources, most notably the IQlight® company. All descriptions of this remarkable system, including those by the designer himself, interpret the constructs in terms of quadrilaterals (indeed "IQ" stands for "Interlocking Quadrilaterals") or rhombuses (the single piece,

[1] The bifastigium, as defined above, is not usually considered a polyhedron since it has two pairs of coplanar triangular faces. This does not matter here since we are interested only in its associated graph.

Figure 7 : *Icosidodecahedron + rhombic triacontahedron (60 elements), Rhombicosidodecahedron + deltoidal hexecontahedron (120 elements)*

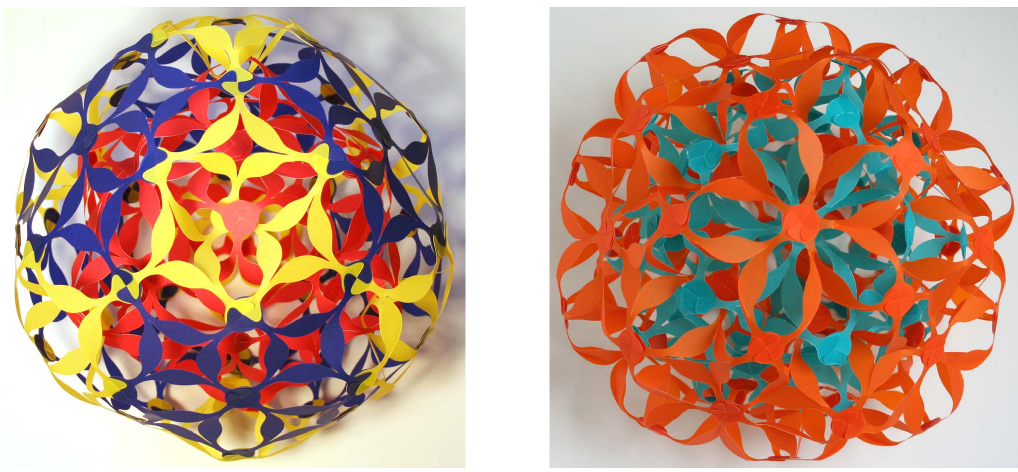

Figure 8 : *Two superimposed models with the same symmetry: rhombicuboctahedron / cuboctahedron, icosidodecahedron / icosahedron*

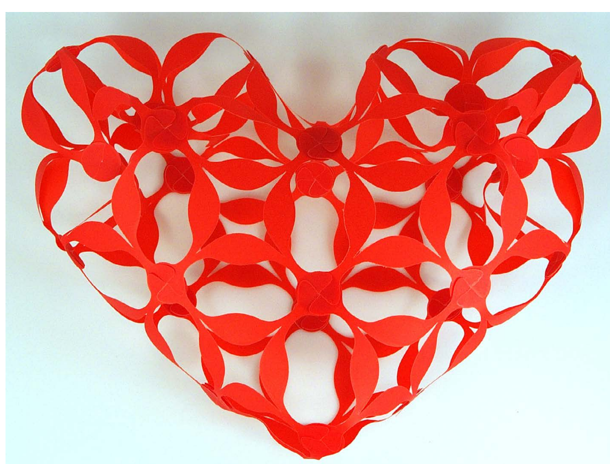

Figure 9 : *Three adjacent cubes in an L formation (28 elements)*

Figure 10 : *An IQlight® piece, the 30-piece model, some other models*

shown in Figure 10, is essentially a rhombus with corners at the ends of the four notches). The iconic 30-piece lamp in Figure 10 is always described as a model of the *rhombic triacontahedron*, which has 30 congruent rhombic faces and is the dual of the *icosidodecahedron*. While this is a good description of this particular lamp, other objects, in this view, are models of some complex polyhedra with non-congruent quadrilateral faces which are not easy to find. Another interpretation of the IQlight® models is obtained by viewing each piece not as a rhombic face but as two dual edges (the diagonals of the rhombus), as described in this paper. In this view, the 30-piece lamp is a model of the dodecahedron-icosahedron compound which have 30 edges each and the 32 connection nodes in the model represent the vertices and faces of the dodecahedron and icosahedron in this compound. (This is, essentially, the same model as the one shown in Figure 5.) Other IQlight® lamps can be similarly described in terms of well-known, simple polyhedra and new models can be built from any polyhedron and its dual (subject to physical limitations).

Unlike the IQlight® system, where the exact geometry and physical characteristics of the plastic pieces are essential, the more general system described here allows for a large variety of shapes and materials which can be chosen to achieve many artistic effects. To allow us to build the simple Stella octangula model (which cannot be, physically, built with the IQlight® pieces) we based the construction elements on squares, not rhombuses. In other words, the two modeled edges (the square diagonals) have the same length before bending them when building a model.

Workshop

In the workshop, we will use pieces that are easy to connect and build a few models starting with the most simple ones; we will provide enough pieces for all participants to take home the objects they build. We will also discuss other methods of implementing the modeling method presented in this paper.

References

[1] M. D. C. D. Ioana Browne, Michael Browne and C. Ionescu. A fun approach to teaching geometry and inspiring creativity. In G. W. Hart and R. Sarhangi, editors, *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture*, pages 587–592, Phoenix, Arizona, 2013. Tessellations Publishing. Available online at `http://archive.bridgesmathart.org/2013/bridges2013-587.html`.

[2] R. Roelofs. The concept of elevation applied to flat tiling patterns. In D. M. Kelly Delp, Craig S. Kaplan and R. Sarhangi, editors, *Proceedings of Bridges 2015: Mathematics, Music, Art, Architecture, Culture*, pages 9–16, Phoenix, Arizona, 2015. Tessellations Publishing. Available online at `http://archive.bridgesmathart.org/2015/bridges2015-9.html`.