Fractal Flipbooks

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Abstract

Riffing off our earlier work using flipbooks to teach about transformations of polyhedra, we present an exercise in which students develop flipbooks that illustrate multiple ways of generating the Sierpinski triangle fractal. Creating and experimenting with the flipbooks enables students to discover and review how different generative processes can result in the same fractals.

1 Introduction

In this paper, we describe a flipbook exercise in which students actively discover fractals by iterating a basic construction operation to draw higher and higher-order fractals on sequential flipbook pages. We provide students with a starting order-0 fractal figure and an operation. After completing our proposed workshop, students should have a sense of how fractals are self-similar. Additionally, they should understand that self-similarity allows the same fractal to be generated in multiple ways.

The approach we describe here riffs off of (and borrows basic flipbook construction instructions from) our 2013 workshop on "Flipbook Polyhedra" [7], which used flipbooks to help students visualize relationships between polyhedra that share symmetry groups. In our "Flipbook Polyhedra" exercise, students constructed animated flipbooks displaying progressive truncations of regular polygons and polyhedra; these flipbooks illustrate that regular polygons are self-dual, and that the cube and octahedron share rotational and reflectional symmetries. Here, just as in [7], the process of developing and flipping flip books helps students gain intuition for subtle geometry. Unlike in our polyhedron exercise, however, the key educational insights here occur across related flipbooks, rather than within a single flipbook: by developing multiple flipbooks with the same ending fractal, students learn that there are multiple paths to the same mathematical outcome.

The remainder of this paper is organized as follows: First, we survey the existing literature on flipbooks and flipbook visualization. Then, we present our workshop exercise, and explain its mathematical significance. Finally, we describe several extensions.

2 Related Work

Flipbooks have been used for around a century and a half primarily for entertainment; they were first patented in 1868 under British Patent No. 925 (see [2]). Recently, several authors have advanced approaches to using flipbooks for math and science instruction (see, e.g., [4] and [5] for applications to arithmetic and chemistry instruction, respectively). To our knowledge, however, our series of flipbook workshops ([7] and the present paper) are the first to use animated flipbooks for teaching geometric concepts. Additionally, *flipbook-style visualization*—sequencing of gradually varying images—has been used to illustrate numerous concepts in applied mathematics (see, e.g., [1], [3], [9], [10], [11], and [12]).[1]

[1]The preceding references use flipbook-style visualization to illustrate topics ranging from surface morphing to social networks; for a more comprehensive discussion, see [7].

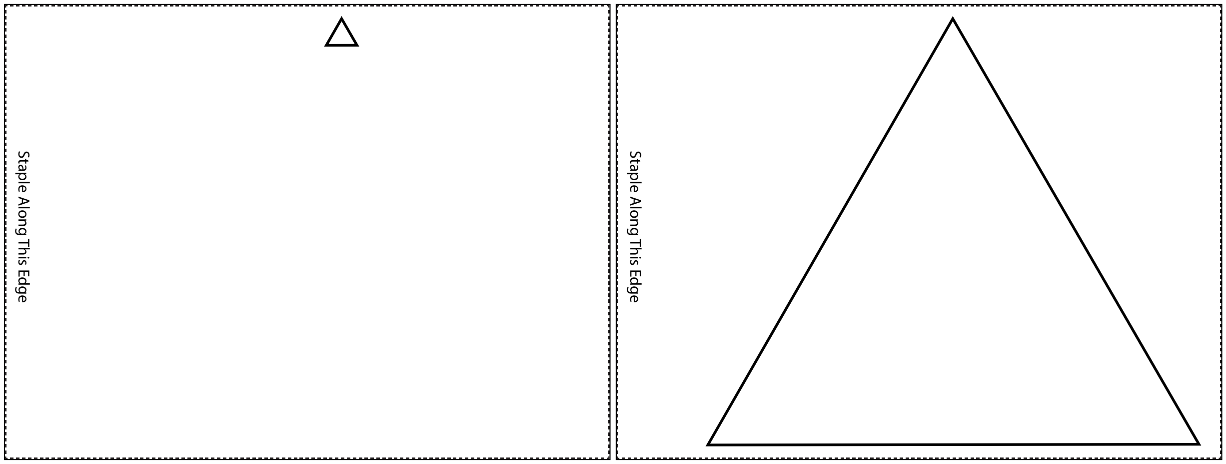


Figure 1 : *Two templates to be used for the flipbooks' final pages.*

Our fractal flipbook exercise allows students to construct and then regenerate their own fractals, step-by-step. As Hanson *et al.* [6], Hundhausen *et al.* [8], and Badame and Boothe [1] have pointed out, such interactive visualizations are especially powerful for improving student understanding and intuition.

3 Workshop Description

We describe an example fractal flipbook exercise using the template sheets provided in Figure 1: a small triangle for the first method of generating a Sierpinski triangle and a large triangle for the second method of generating a Sierpinski triangle.

3.1 Setup and Materials

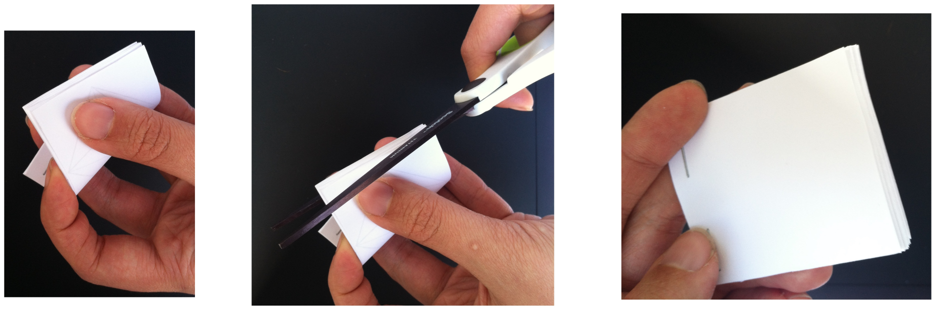
The setup in our exercise is follows that of [7], but uses different templates. Each student should either make or be provided bound flipbooks prepared according to the instructions below. Students will also need a pencil, a pen, and an eraser. They may also use markers or colored pencils, for coloring in their flipbooks. Straight edges may be helpful, but are not strictly necessary.

3.2 Procedure

Flipbooks measure 6cm by 8cm. Each flipbook is composed of eleven sheets of blank white paper, which are followed by a template sheet that serves as the basis of the exercise. Example template sheets for Sierpinski triangles are presented in Figure 1.

For the mechanics of physically building flipbooks, we exactly follow the construction of [7]: Flipbooks should be stapled together along one edge. To aid in flipping, it is helpful, but not strictly necessary, to make a bias cut along the edge opposite the binding. To make a bias-cut, bend the flipbook into a U-shape, as shown in Figure 2a. Then, make a straight cut as shown in Figure 2b. This will result in a bias-cut flipping edge as pictured in Figure 2c. Note that a bias-cut flipping edge will flip more easily than a straight-cut flipping edge, but will flip in only one direction. It is preferable to wait to bias-cut the edge until after a decision has been made on the flipping direction.

For simplicity, as in [7], we refer to pages in the flipbooks by number, with page 12 (the last page in the flipbook) being the template page. The fractal generation should span pages 3–12; flipbooks' front pages



a b c

Figure 2 : *Method for creating a bias-cut flipping edge ([7]).*

may be used for title and byline. We assume that each student will have time to complete both flipbooks, but depending on time constraints you may wish to split students into two groups with half completing each flipbook variation.

3.2.1 Small Triangle Template

Students begin by creating Sierpinski triangle flipbooks using the small triangle template. The procedure is as follows:

1. On page 11, students should trace the triangle on page 12.
2. On page 10, students should trace the triangle from page 11, and then add two identical small triangles such that the top vertex of the new triangles correspond to either the bottom left and bottom right vertices of the original triangle as in Figure 3.
3. On page 9, students should trace the design from page 10.
4. At this point, the procedure should be self-similar for pages 3–8. On each even-numbered page, students should trace the design from the last page drawn and then add two new copies of that design, such that the tops of the new copies correspond to the bottom-left and bottom-right vertices of the original design as in Figure 3. On odd-numbered pages, students should simply trace the last page drawn.
5. Finally, students should color all upward-facing triangles.

3.2.2 Large Triangle Template

Students next create a flipbook using the large triangle template. The procedure for this second flipbook is as follows:

1. On page 11, students should trace the triangle on page 12.
2. On page 10, students should trace the triangle from page 11, and then find the midpoints of all edges of that triangle and connect them, forming a downward-facing triangle.
3. On page 9, students should trace the design from page 10.
4. At this point, the procedure should be self-similar. On even-numbered pages, students should trace the last page drawn and then add downward-facing triangles to each upward-facing triangle. On odd-numbered pages, students should simply trace the last page drawn. For this design, students can choose to leave the first two pages blank for a title, or to continue and draw an even higher-order fractal.
5. Finally, students color all upward-facing triangles.

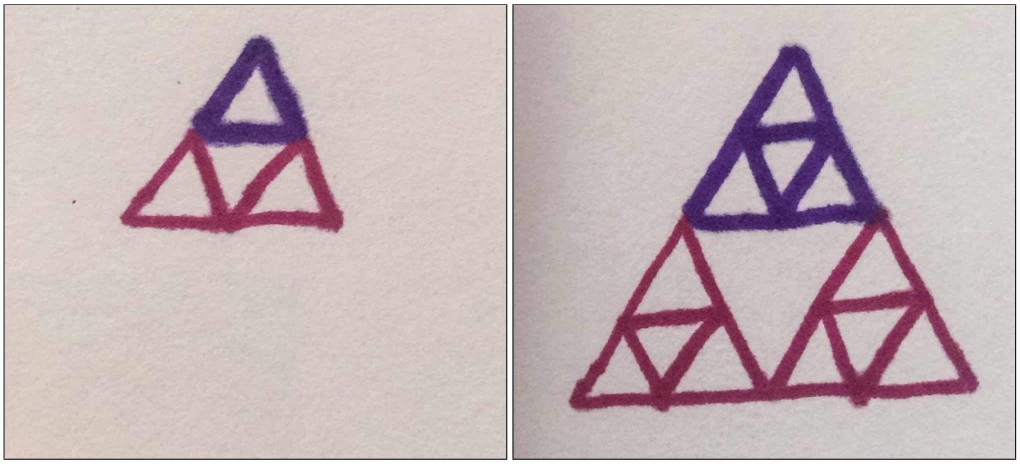


Figure 3 : *The resulting drawings on page 10 (left) and 12 (right) of the Small Triangle flipbook. The red indicates the new additions between those steps and the earlier ones.*

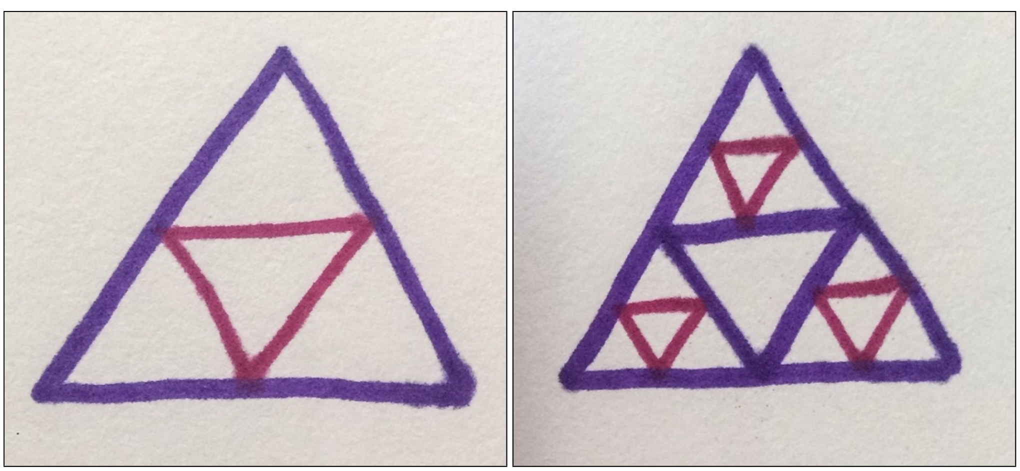


Figure 4 : *The resulting drawings on page 10 (left) and 12 (right) of the Large Triangle flipbook. The red indicates the new additions between those steps and the preceding ones.*

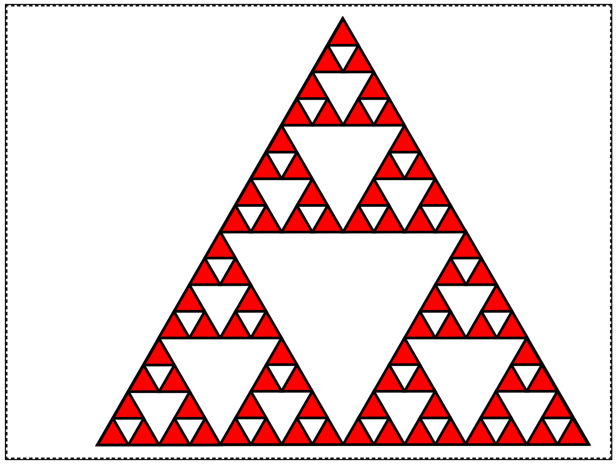


Figure 5 : *The level-4 Sierpinski triangle resulting as the completion of both flipbooks.*

3.3 Final Results

After finishing the two flipbooks, students should discover that each ends with the same figure: a level-4 Sierpinski triangle, as pictured in Figure 5. As a small additional exercise, students may remove the staple from one flipbook, reverse the pages, and prepend them to the other flipbook to generate a double length flipbook. This longer flipbook may appear to grow the fractal and zoom out (or vice versa).

4 Discussion

Through our flipbook exercise, students should begin to develop an understanding of the self-similarity of fractals—repeatedly interating the same steps generates the final result.

The workshop highlights two different ways of generating the Sierpinski triangle fractal. The small triangle strategy can be thought of as growing an ever larger fractal from the base case, while the large triangle strategy might be thought of as zooming in to see the repeating details of the fractal. There are many other methods for generating the Sierpinski triangles, and it might be instructive to discuss them after this exercise.

There is quite a lot of drawing required to complete this workshop. This is both unavoidable and intentional. The goal is for students to physically notice how the number of triangles they must draw increases as they draw higher and higher levels of the fractal.

Interesting basic follow-on exercises might include predicting the number of small triangles at each iteration (or the number of triangles of each size), and computing the total perimeter (or total area). With more time, it would be fun to encourage students to try using similar procedures on different initial shapes, so as to produce custom fractal flipbooks.[2] A follow-on exercise for a more advanced class would be to calculate the Hausdorff dimension of the Sierpinski triangle.

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