Rhombic Triacontahedron Puzzle

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Abstract

This is a hands-on cardboard construction workshop that uses the geometry of the rhombic triacontahedron to pose a series of interesting challenges. Participants will first create a paper model to understand the basic structure of the shape and how it can be dissected into twenty rhombohedra. They will then build a large-scale rhombic triacontahedron out of cardboard rhombi, adding a combinatoric coloring challenge. The final result is a beautiful structure both mathematically and visually. We have led this *Making Math Visible* workshop at the university and high school level, and have observed that students enjoy creating and solving a puzzle that is also an *objet d'art.*

Introduction

Over the past year, we have been creating a series of workshops for teachers and students that present mathematical ideas through a hands-on, visual approach. Our goal is to encourage teachers to inspire students by making math more visible in the classroom. We have observed that students become deeply connected with math when they take an active part in building large-scale, visually engaging objects that can be left on display as a piece of art. Their interest in mathematical ideas increases as questions naturally arise during the construction. Students take ownership of their creation and are able to explain it to their peers and parents, drawing the community into a dialog about mathematics.

Figure 1: *Rhombic Triacontahedron puzzle made of stained plywood.*

In field-testing these workshops, we have observed great enthusiasm from teachers and students. We have been writing up lesson plans and publishing them online in order to make them freely available to educators everywhere around the world. Our goal is to produce enough activities for teachers to have a year's worth of material which they may use weekly in class or in an after-school math club. We thereby hope to encourage teachers to foster a strong math culture in their schools and communities by making the beauty of mathematics visible. For details, readers are referred to http://makingmathvisible.com. [1]

Our general approach often starts with building a small model that provides a mental blueprint for a human-size construction that follows. Working with scissors, paper, and tape is a useful cost-effective way to introduce many topics in three-dimensional geometry. It helps students understand important spatial relationships using materials and processes they are comfortable with. Continuing with an oversized version of the now-familiar shape invites a physical interaction that solidifies the concepts in students' minds. The sculptural nature of the large-scale object elevates the mathematical ideas to a higher level of artistry. See Figure 1. Cardboard and laser-cut plywood lend themselves well to making affordable large-scale 3D objects. In addition, the collaborative aspect of this type of construction encourages participants to engage in authentic conversations about math.

For this Bridges Conference workshop, we have chosen one of our favorite constructions. It exemplifies a natural blending of deep mathematical ideas combined with the pleasure and physicality of a 3D construction, the joy of puzzle solving, and the aesthetic satisfaction of a sculpture.

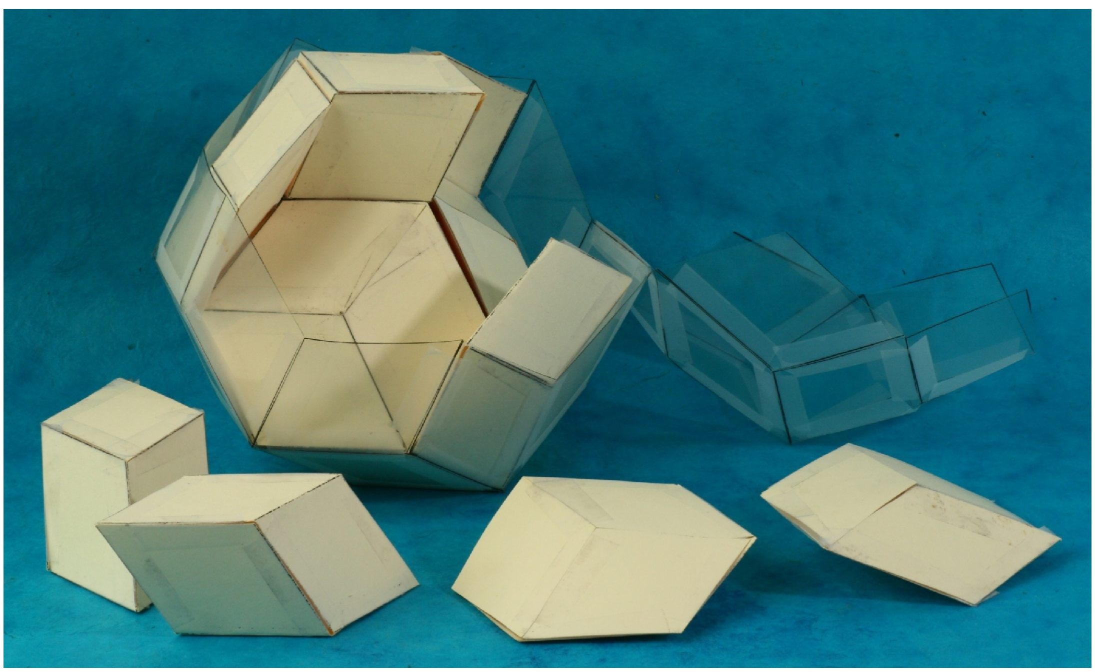

Figure 2: *Rhombic Triacontahedron puzzle made of monochromatic paper.*

Mathematical Background

The rhombic triacontahedron (RT) was first discovered in 1625 by the German mathematician and astronomer Johannes Kepler. [2] In 1865 it was understood as an Archimedean dual polyhedron when the Belgian mathematician Eugene Catalan placed it in a larger polyhedral system. [3] The RT is the dual to the icosidodecahedron and is bounded by thirty congruent golden rhombi. The ratio of the diagonals of the rhombus is the golden ratio, approximately 1.61803.

In 1938, the German mathematician Gerhard Kowalewski published a book, *Der Keplersche Korper und andere Bauspiele,* which describes how the RT can be dissected into twenty rhombohedra, ten acute and ten obtuse, all with the same rhombic faces. [4] He also introduced a coloring aspect to the dissection which adds another layer of interest. Until its partial translation by David Booth in 2001, Kowalewski's discovery was only known in the English speaking world through a small footnote in Coxeter's 1963 book, *Regular Polytopes* and a 1996 web page by Hart. [5, 6, 7]

One of the interesting aspects of the RT geometry is that it is a zonohedron with six directions of edges. [8] Each rhombus of its surface is generated by two of the six directions. When it is dissected, each of the twenty interior pieces is generated by three of the six directions. It is combinatorially complete in the sense that all twenty ways of choosing three out of six appears once. As a consequence, the RT is a great vehicle for introducing combinatorics to students in an age-appropriate, comprehensible, hands-on manner.

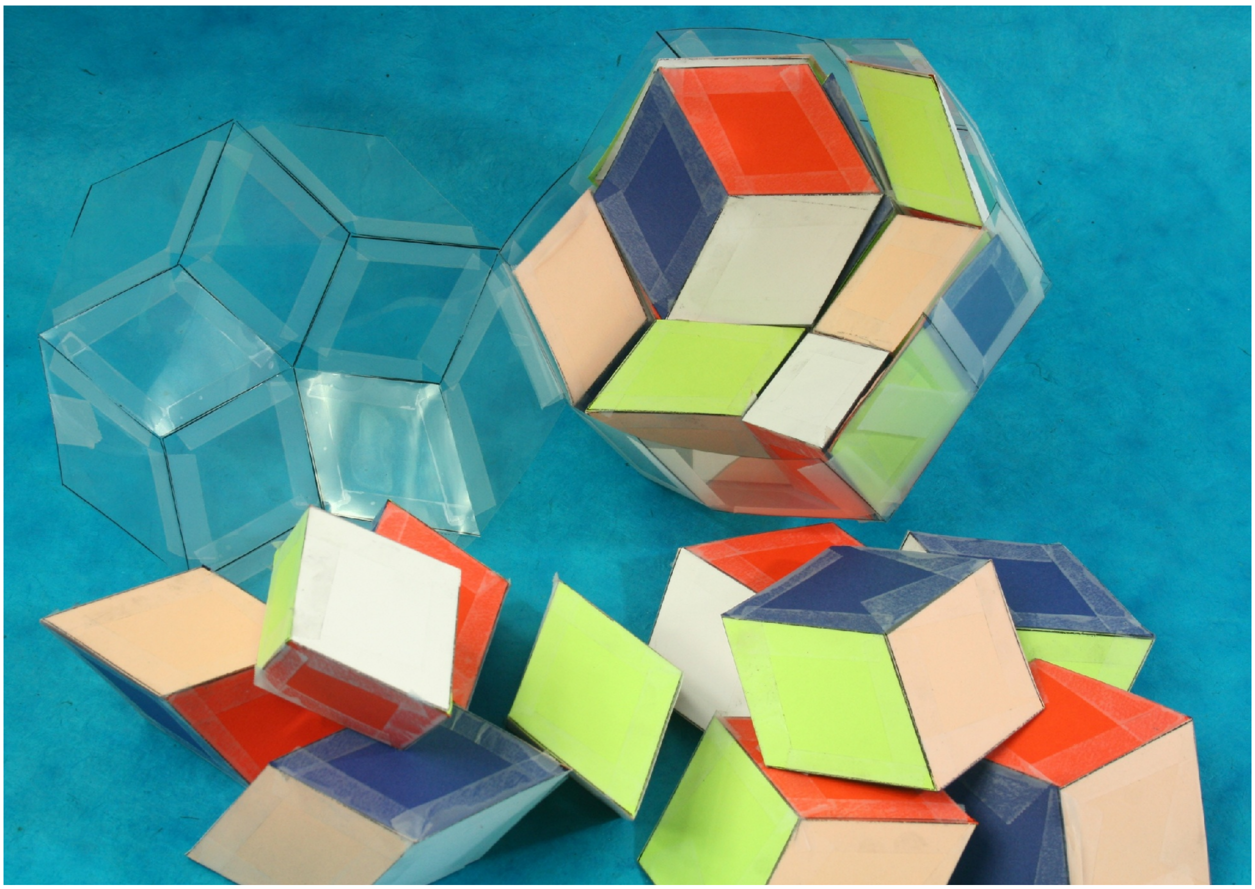

Figure 3: *Rhombic Triacontahedron puzzle made of colored paper.*

Construction Details

We have made the RT puzzle in paper, cardboard, and wood. The paper versions, shown in Figures 2 and 3, are great teaching tools that can be built with little preparation. They only require printing or photocopying, cutting, and taping. However, the result is less permanent and more easily damaged. The wood version is a very robust, beautiful object, made of laser-cut wooden rhombi that have been beveled, stained, and glued together. Although this version requires more planning and the use of a wood shop, it results in a lovely geometric model that can be used in a classroom for the long term. See Figures 1 and

6. The large cardboard RT, shown in Figure 4, combines the pedagogy of the paper version with the sculptural characteristics of the wood version. We provide templates that can be used on a laser-cutter to produce rhombi with short finger joints that ensure the parts mesh perfectly. [1] This in turn, guarantees the dihedral angles are correct, so the blocks assemble with satisfying precision. We envision that laser cutters will be found in increasingly many schools in the near future. There are also service bureaus which can cut the parts. Even without access to a laser-cutter, the parts can still be produced with a saw.

We have divided the workshop into three sections, described below. Each section is a lesson in itself which follows a three-part pedagogical model: a minds-on activity, a hands-on activity, and a conclusion/consolidation. There are also optional extensions that suggest questions for students to delve deeper into the subject. The full details are available in the *Making Math Visible* reference. [1] Given the time constraints at the Bridges Conference, we will present a combination of various aspects of the three sections, culminating in a large cardboard construction which will be displayed at the venue.

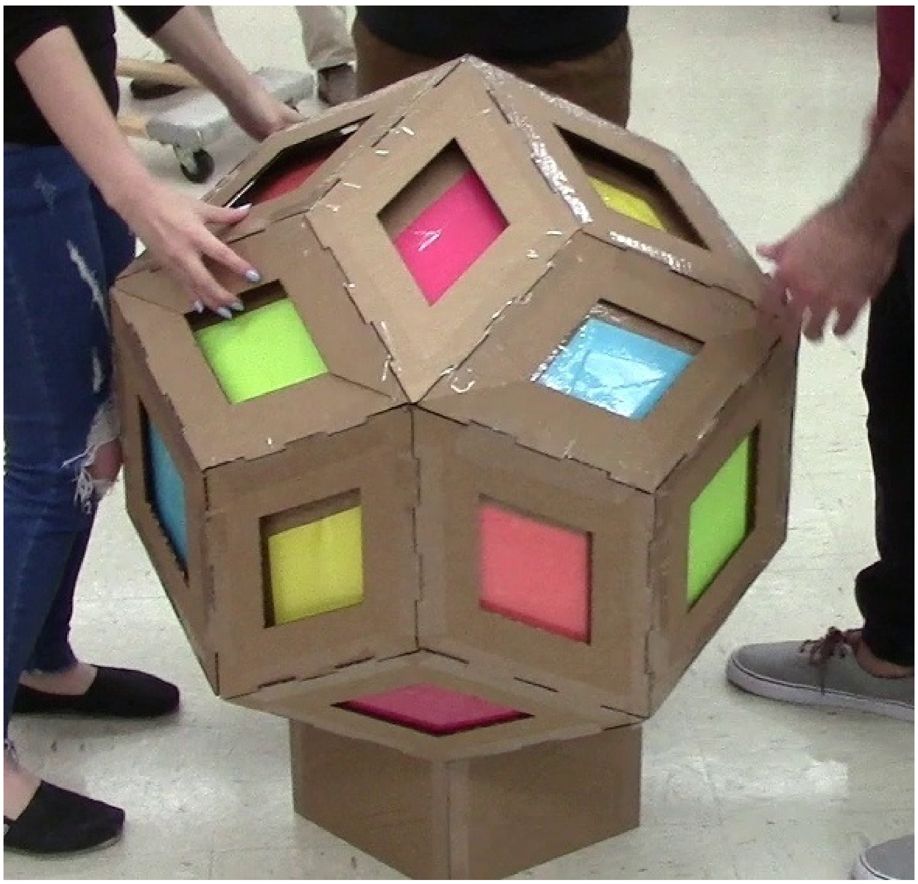

Figure 4: *Rhombic Triacontahedron puzzle made of cardboard.*

Section 1: Paper Version. We begin with the construction of twenty paper rhombohedra to familiarize students with the underlying geometry of the RT's components. Students can play and experiment with these blocks, exploring how they fit together in interesting ways. The RT is a natural consequence which students discover as they play, but it requires many hands to hold the twenty blocks in place. The need for a container becomes apparent and students use clear acetate rhombi to construct an RT-shaped shell. The result is a beautiful puzzle that students can solve by filling the shell with the rhombohedra, as seen in Figure 2.

This hands-on activity is book-ended by a minds-on drawing exercise which illustrates the use of parallel lines in arbitrary directions to generate zonohedra and by a consolidation that emphasizes the discovery of "belts" which share a parallel direction.

Section 2: Colored Paper Version. The second section of the overall workshop has the same structure as the first section, but introduces five colors of paper and specific coloring constraints. Each rhombohedron is made with three of the five colors, with opposite pairs of faces matching, resulting in twenty unique components. A color-matching rule is introduced: when assembling blocks, adjacent faces must be the same color. The geometry is the same, yet the puzzle challenge is at a much higher level of difficulty. See Figure 3.

This hands-on activity is book-ended by a minds-on introduction to Pascal's triangle and the *n*-choose-*m* function and by a consolidation in which students come to understand the combinatorics of the edge directions appearing in the surface rhombi and the interior rhombohedra.

Figure 5: *Assembling the cardboard Rhombic Triacontahedron puzzle.*

Section 3: Cardboard Version. In the third section of the workshop, students come together to construct a large cardboard RT dissection puzzle, shown in Figures 4 and 5. This starts with an uncolored model that is purely a geometric puzzle. Then the faces are colored to complete the components. The impressive construction that results remains as a fixture in the classroom embodying the joy and beauty of mathematics.

This hands-on activity can be book-ended by reviewing any of the minds-on introductions and consolidations of the previous sections. We leave it up to teachers to select the appropriate material.

Conclusions

The rhombic triacontahedron dissection workshop is a great example of how to combine multi-leveled introduction of mathematical concepts, collaborative engagement, and an aesthetic sculptural sensibility. We have conducted the workshop with high school students in a Bronx, New York City public school and with community college students in Jacksonville, Florida. In both cases, students were enthusiastic to discover a rich connection between mathematics and art. The construction process evoked questions and interest from students who typically don't have a strong affinity to mathematics. We are encouraged and convinced that hands-on workshops of this kind can change the mindset and confidence of students and improve their relationship to mathematics.

Teachers have told us that they want to incorporate more of these types of activities into their teaching practice, having directly observed a change in attitude in their students. A selection of direct quotes from teachers and students can be found on our MakingMathVisible website. [1] Our future work is to continue developing and documenting lesson plans, providing teachers and students with additional workshop ideas.

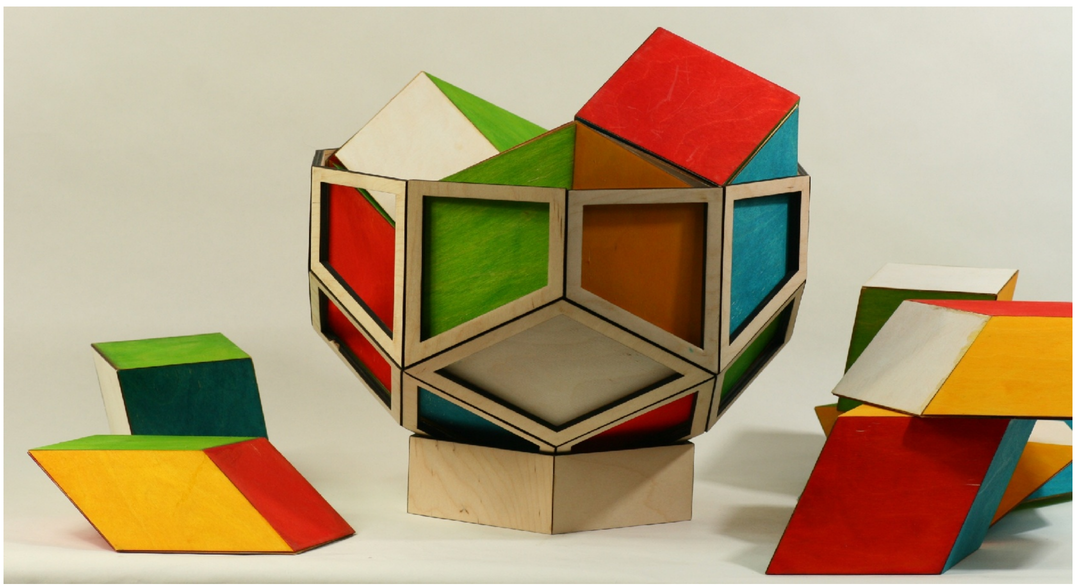

Figure 6: *Wood Rhombic Triacontahedron puzzle.*

References

[1] Hart, G. and E. Heathfield, http://makingmathvisible.com

[2] Kepler, Johannes, *The Harmony of the World,* 1625, (transl. E.J. Aiton, A.M. Duncan, and J.V. Field, 1997, American Philosophical Society).

[3] Catalan, Eugene, "Mémoire sur la Théorie des Polyèdres," *J. l'École Polytechnique,* 41, 1-71, 1865.

[4] Kowalewski, Gerhard, *Der Keplersche Korper und andere Bauspiele,* Koehlers, Leipzig, 1938.

[5] Coxeter, H.S.M., *Regular Polytopes,* Macmillan, 1963, (Dover reprint, 1973).

[6] English translation of [4] as *Construction Games with Kepler's Solid,* transl. David Booth, Parker Courtney Press, 2001.

[7] Hart, G., http://www.georgehart.com/virtual-polyhedra/dissection-rt.html

[8] Hart, G., "A Color-Matching Dissection of the Rhombic Enneacontahedron", *Symmetry,* vol. 11, 2000, pp. 183-199.