Scales and Temperament from the Mathematical Viewpoint

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Abstract

Scales constructed from pure harmonic ratios often contain *enharmonic* note pairs that do not sound in tune when played together. We examine the mathematics behind *equal tempered* scales that avoid this problem by insisting that the harmonic ratios between any pair of successive notes be identical. The mathematical problem is to construct equal tempered scales that do a good job of approximating the notes of scales built from pure harmonic ratios. Particular old and new solutions to these problems are discussed herein.

Introduction

A musical note is, roughly, any sound with a well-defined *pitch.* In practice, musical notes are complex collections of many different frequencies. However, typically a single frequency emerges as dominant. Human ears seem to seek out a dominant pitch, even when one is not there. Additional frequencies then add character or "color" to the note. Indeed, in practice, a pure pitch has a flat one-dimensional sound. In the discussion that follows, we focus only on the dominant pitch of each note.

The set of notes used in a melody forms the *scale* (or *scales*) out of which that melody is built. Our concern is with notes in a relative sense. In particular, we dispense with talking about absolute pitch, the specific frequency of vibration, and instead focus on relative pitch, the ratio between the frequencies of a pair of vibrating objects. For example, for many a standard A has a frequency of 440 Hz. Doubling the frequency to 880 Hz produces another A, as to western ears these two frequencies represent the same note. These two pitches have a relative ratio of 2/1. More generally, relative pitches, or *intervals* as they are called in music theory, are the ratios between the frequencies, and are naturally enough in 1-to-1 correspondence with the positive real numbers. Historically, attention has been paid to whether intervals are integral, rational or irrational, with the rational intervals occurring most naturally in *scales*, each of which are associated to a specific root note of the scale called its *key*. Harmonicas and bagpipes are examples of instruments that only have enough notes to play in one or two keys. Indeed, concertinas and accordions can be thought of as several harmonicas "in parallel" achieving the ability to play in a wider variety of keys by supplying new notes.

The most common approach to scale construction is to fix the notes obtained via the doubling of frequency and fill in some number of intermediate notes. For example, in the standard western tradition the doubling of frequencies is an "octave", i.e., the scale is a heptatonic one with seven notes, the familiar *do, re, mi, fa, sol, la, ti,* returning to *do*. Though many of our examined scales will have fewer or more notes, we will as in common musical practice refer to the doubled frequency note as the *octave* of a given root. In western music we commonly use pentatonic and onatonic scales, i.e., the familiar blues scale and the 12 frets or keys to an octave found in guitars, banjos, mandolins, ukuleles, and pianos. This latter system allows the formation of "half steps" which then allow us to play the various sharps and flats required for scales in different keys. However, as we shall see, adding these extra notes raises surprisingly complex and subtle challenges. The general mathematical problem underlying these considerations is to describe the construction of a scale containing each of a given list of intervals or reasonable approximations of them.

Pythagorean and Just Tunings

One particularly simple answer to this question is commonly attributed to the Pythagorean School, but was also used throughout Europe into the sixteenth century [1]. Pythagorean scales are generated by a single pure interval. The simplest non-trivial harmonic ratios are 3/2 (the perfect fifth) and 2/3, which normalizes (i.e., scaled by the appropriate power of two to lie in the "fundamental" octave of intervals between 1 and 2) to 4/3 (the perfect fourth). As intervals, these ratios are closely related. Move up from the root by 3/2 and then by 4/3 and the note reached is 2/1, or one octave.

Starting with a root, moving up two perfect fifths produces a note that normalizes to 9/8 from the root. This observation gives us one way to construct a sequence of notes from pure ratios. In general, the ratio of the n^{th} note of this sequence from the root note has a simple formula involving $\log_2(3)$. In this process, the normalized octave is noticeably sharp. Its normalized value is s approximately 1.0136. This ratio, now known as the *Pythagorean Comma*, presents a serious challenge for the Pythagorean system. Insisting that each new note be related by a perfect fifth to some previous note in our scale does not ensure that arbitrary pairs of notes have any kind of nice harmonic relationship. It also fails to ever generate a perfect octave. One historical solution to this problem was to *temper* the Pythagorean scale by flattening the twelfth ratio into a perfect octave. In general, a *temperament* or *tempered scale* is any scale that results from adjusting the intervals of some other scale.

One benefit of the Pythagorean method in general is that it allows us to iteratively refine the scale if desired. For example, as a way to better codify the music of his day [1], the great 13th century Persian musician Safi al-Din al-Urmaw used a similar technique to construct a 16 note scale, specifically, a scale that begins at B and increases by perfect fourths (i.e., a factor of 4/3) at each stage:

B, E, A, D, G, C, F, Bb, Eb, Ab, Db, Gb, Bbb, Ebb , Dbb , Gbb.

These types of Pythagorean scales are not just of historical interest, but are in fact employed by contemporary American musicians such as Mavis Staples and David Lindley, working initially by ear and heavily influenced by their experience with fretless instruments and the human voice, who incorporate these scales into their instruments and performances. For comparison, Figure 1 contains images of a standardly fretted "diddley bow" belonging to one of the authors and a uniquely fretted lute belonging to Mr. Lindley and used in his performances.

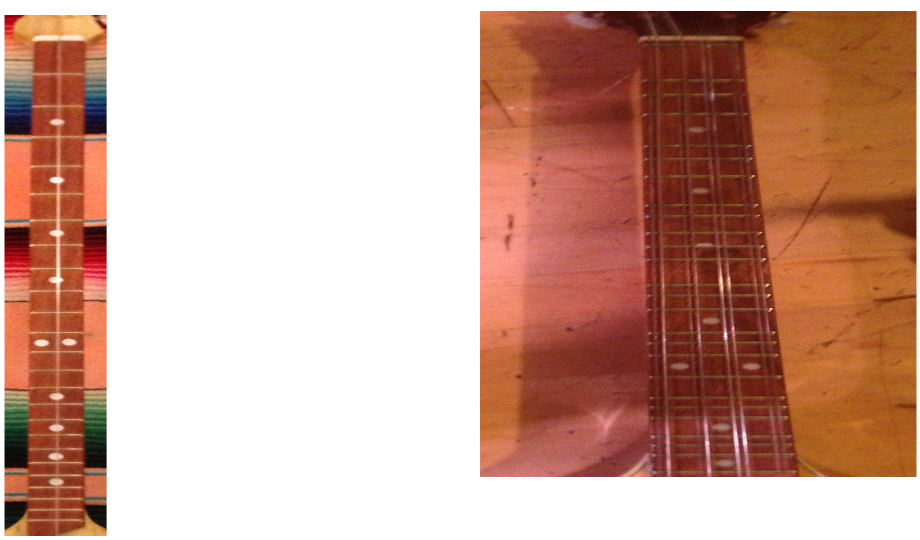

Figure 1: *Standard and "non-standard" scale patterns*

Fretted western stringed instruments standardly contain 12 frets per octave. In Figure 1, we see the octave marked by a pair of inlayed dots in each image. On the left, we see the standard 12 frets per octave. On the right, there are 18 frets in the first octave, six additional frets have been added to a standardly fretted lute, to yield a modified Persian scale.

"Just" scales are constructed analogously to Pythagorean ones, but use more pure intervals as generators [1]. Here we obtain a multi dimensional lattice of notes related by pure intervals. For example, we can build a system that includes the perfect fifth 3/2 and perfect major third interval 5/4. We carry out the same construction as in the Pythagorean case, but now we get a two-dimensional lattice structure. This extra complexity has a big benefit. For example, in such a scale it is possible to play 3 note chords in which each pair of notes is related by a pure harmonic ratio. On the other hand, the problem of the Pythagorean comma is also amplified in such a tuning system, and so such systems require more tempering.

Equal Temperament

Looking at the Just and Pythagorean scales constructed above, it is evident that their notes are unevenly distributed across the octave. This has a profound consequence that we have yet to address. In the above examples, all of our scales were built with reference to a root note. This is because otherwise we would be normalizing our sequence of fifths with respect to a different interval. Thus, to play a melodic figure using A as the root note and then to repeat that melodic figure using B as the root note, we would need two separate scales. This is evident when comparing the ratios of the first to the fifth and second to the sixth in any given key of the Pythagorean and Just scales. The two ratios should be equal (indeed, 3/2) but are not. In a single piece of music, moving from one root note (called a *key center*) to another is called *modulation*. As classical composers began exploring modulation, it became increasingly practical to simplify the bewildering number of notes available. This led many composers to conclude that the intervals of the scale ought to be identical no matter which octave or key they are reduced relative to.

To accomplish this equalization, each pair of sequential notes in the scale must have the same ratio as any other. This leaves only one possibility for scales with n notes, ratios of precisely $2^{1/n}$, the so-called *n-ET* scale. Thus for a twelve note scale that can support modulation, the intervals between successive notes must equal $2^{1/12}$. Since this number is irrational, we never achieve the kind of pure intervals sought after in the Pythagorean and Just systems. On the other hand, certain equal tempered scales approximate these other scales reasonably well. For example, in the 12-ET, the fifth note of the major scale has a ratio to the root of $2^{7/12}$, an interval of about 1.498, a fairly close approximation to 3/2. This brings up a natural *goodness of fit* question, i.e., precisely how well does a division of the octave into n equally spaced notes approximates a given set of Just intervals ? A useful goodness of fit measure also has weights assigned to each target ratio reflecting the relative importance of the notes represented, e.g. a perfect fifth might be assigned a weight of 10 and the minor third assigned a weight of 4. This requirement mimics the historical considerations of past practitioners, who labored mightily to produce the requisite numerical data.

The goodness of fit issue explains why 12 is a particularly good choice for the number of notes in an octave [2]. As per [3], one can approximate the irrational $\mathrm{Log}_2(3)$ via an additive continued fraction of form [1;1,1,2,2,3,1,5, ...]. Stopping after four layers we get the value of 19/12 as a rational approximation, saying that $2^{19/12}$ is approximately 3 and hence that $2^{7/12}$ closely approximates 3/2, our perfect fifth, as we saw above. One checks that the perfect fourth and the major and minor thirds are similarly well approximated. An exception here is the seventh note, which is somewhat sharp. This sharpness induces "tension" in the seventh chord and is one reason why seventh chords appear so frequently in traditional blues. For other "well fitting" ETs, we can go another layer down the continued fraction to get an even better approximation of 65/41, suggesting an equal tempered scale of 41 notes, the 24^{th} of which would be very close to a perfect fifth. Indeed, on the internet one can easily find contemporary music written and performed in the 41-ET and its extension, the 205-ET employed in the Tonal Plexus. Historically, various features of the goodness of fit of the 53-ET to various Just and

Pythagorean scales appears in the work of Ching Fang (78–37 BC), Nicholas Mercator (c. 1620–1687) and even in unpublished work of Isaac Newton [4].

To use today's computational technology to explore these issues, the authors produced a JAVA applet [5] to produce the numerical data necessary to the investigation of equal tempered scales and their goodness of fit to various Just and Pythagorean scales. Using such the authors were easily able to rediscover the 31, 41, 43, and 53-ETs familiar to music theorists, and to uncover previously unobserved "well fitting" 65 and 125-ETs, the 65-ET having particularly well fitting sixth and seventh notes to a Just 12 note scale when compared to the 12-ET, as shown in Figure 2. Of course, the more notes included in a given scale, the more complex that scale is to work with. For this reason, concerns about instrument design also play a big role in the ultimate utility of a scale.

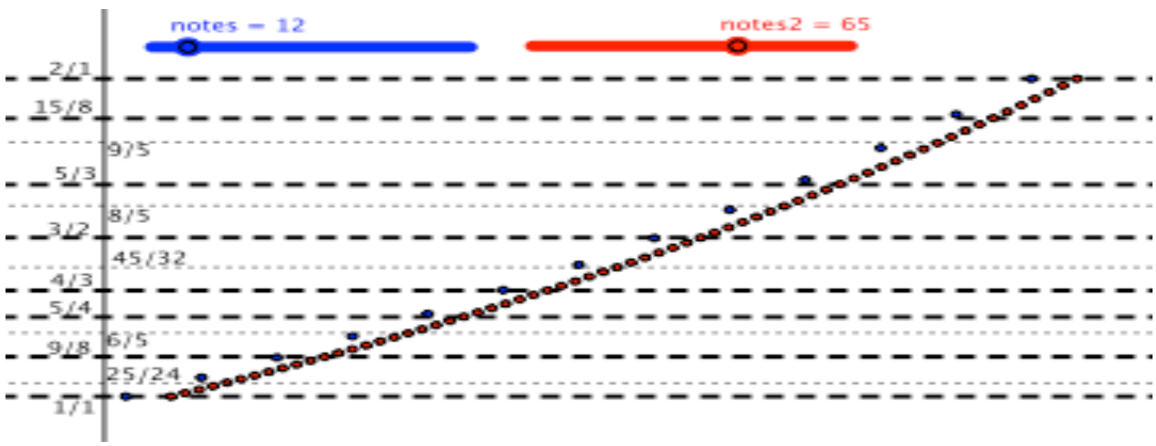

Figure 2: *Comparing the goodness of fit for the 12- and 65-ET scales*

Our investigations indicate that the mathematics behind scale construction is surprisingly rich and may hold a clue as to why, even today, scale construction is an active area of research among music theorists. Modern electronic keyboards can be programmed to simulate any tuning system imaginable. Of course, most music in the West remains firmly rooted in the versatile and ubiquitous 12-ET. Nonetheless, there is plenty of evidence, ranging from the work of contemporary classical composers and that of pop and folk musicians, that western musical traditions are not always faithfully represented in the 12-ET and that the scales and intonations of the other ETs and of the Pythagorean and Just scale traditions continue to have important roles in Western music.

References

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[3] Oskar Perron, *Die Lehre von den Kettenbruchen, Band II*, §13, B.G. Taubner 1957

[4] Wikipedia, *53 equal temperament*, **https://en.wikipedia.org/wiki/53_equal_temperament,** accessed April 21, 2016.

[5] Steve Bleiler and Ewan Kummel, *Exploring equal temperament*, GeoGebra applet available at http://ggbtu.be/m2975305, accessed April 21, 2016.