Teaching and Learning Basic Group Theory through Building Models of Polyhedra

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Abstract

The purpose of this paper is to share the experience of teaching (first author) and learning (second and third authors) basic group theory, as well as other mathematics relevant to the study of symmetry, through building models of polyhedra. An effective and engaging teaching strategy for a first course in abstract algebra is outlined.

Introduction

Students in beginning abstract algebra courses often become frustrated by the barrage of definitions, axioms and properties, which, for instructors, are not easy to motivate and, for students, are even harder to absorb. The situation is not unlike the one in a beginning foreign language course, where a student may have to learn a new alphabet and acquire a substantial vocabulary before learning the language becomes easier and more fun. There may be several factors that motivate and sufficiently excite a student to persevere through this difficult initial stage of learning a new language: an upcoming trip, interest in literature, music and/or art, friends, etc. In abstract algebra courses, however, the task of motivating students often falls largely on the instructor's shoulders. Here, making mathematical models is an invaluable tool. It is a great way to stimulate students' interest in mathematics and turn a mathematics course, that they otherwise might have thought of as unavoidable drudgery, into a fun and enriching experience: "It is really surprising how much enlightenment will come, following the construction of these models rather than preceding it, and once you begin making them you may find that your enthusiasm will grow. You will soon see that each of these solids has a beauty of form that appeals to the eye in much the same way that the abstract mathematics appeals to the mind of a mathematician" (M. Wenninger in the preface to [6]). While "the main use of a model is the pleasure derived from making it" ([1], p. 14), building models and examining the models already made also brings the more abstract notions of mathematics to life and makes them much more interesting, much more accessible to the students. In the Spring, 2014, the first author taught an interdisciplinary honors seminar on polyhedra and symmetry, in which second and third authors were student participants. The present paper summarizes our experiences in this seminar. We also make a few suggestions for improvements.

Making First Models

While several beautiful and innovative ways of making models of polyhedra have been suggested (see, for example, [4], pp. 13–40, or [3], pp. 153–188), we used the traditional approach of making the polygonal faces out of cardstock with tabs around all edges and then assembling the model by gluing the tabs of adjacent faces together. Good general instructions on how to make paper models are given in [6], pp. 12–13, and on Robert Webb's website `http://www.software3d.com/Tips.php`. While mostly following these instructions, we made the modifications below and found them to save time and to simplify the more difficult steps in the construction of models, such as attaching the last face:

- Using a bone folder to score the edges to allow for folding of tabs substantially speeds up the preparation of faces; scoring with a needle is extremely time consuming and tiring
- Using a glue gun was very helpful in constructing more complicated models: it made it easier to reach the tabs in the later stages of the construction and the glue dried much faster than if using the regular glue

We began by making models of the nine regular polyhedra: the five Platonic solids and the four Kepler-Poinsot polyhedra. To visualize these and more complicated polyhedra, we used a wonderful piece of software, called Stella, developed by Robert Webb ([5]; free demo version is sufficient for many purposes). In making the models, we often resisted using printed nets (Stella allows great flexibility in printing nets) even though it would have saved us some time and labor, because having to determine what face has to be connected to what face led to a better understanding of the models we were constructing.

Figure 1: Our first models

Elements of Group Theory and More Models

One of the most remarkable aspects of regular polyhedra is the high degree of symmetry they possess. One can begin analyzing the symmetries of a particular polyhedron by simply trying to list all of them. Already here, it is extremely useful to note that a composition of two symmetries is again a symmetry, as in the following proposition.

Proposition. *A regular tetrahedron has twelve rotational symmetries.*

Proof. It is easy to identify some rotational symmetries of a regular tetrahedron. There are four axes of symmetry, each connecting a vertex to the center of the opposite face, and rotations by $120°$ and $240°$ around these axes clearly rotate the tetrahedron onto itself. Together with the identity (rotation by $0°$, this gives us nine rotational symmetries. It is somewhat harder to see the remaining three rotational symmetries. What happens if we follow one of the eight rotations above by another such rotation? It is sufficient to follow the movement of the vertices. Let us number the vertices and consider two of the above eight rotations, σ and τ, which move the vertices as follows:

$$\sigma: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 4 & 2 \end{array} \quad \text{and} \quad \tau: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 4 \end{array}$$

Following σ with τ results in a symmetry that interchanges vertices 1 and 2, as well as vertices 3 and 4:

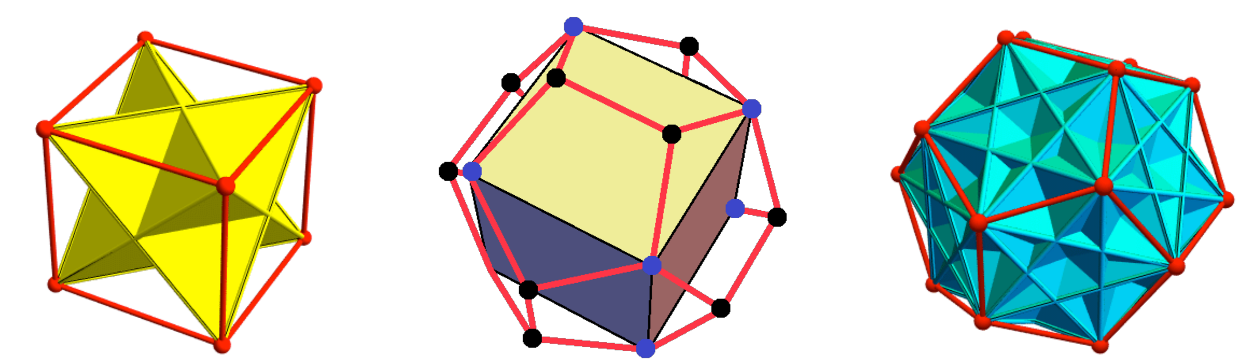
$$\begin{array}{rcccc} & 1 & 2 & 3 & 4 \\ \sigma: & \downarrow & \downarrow & \downarrow & \downarrow \\ & 1 & 3 & 4 & 2 \\ \tau: & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 1 & 4 & 3 \end{array}$$

Thus $\tau \circ \sigma$ (σ followed by τ) is a rotation by $180°$ around the axis through the midpoints of the opposite edges, one connecting vertices 1 and 2, the other, vertices 3 and 4. There are three pairs of opposite edges, hence three such rotational symmetries for a total of twelve. It is easy to see that this is a complete list of rotational symmetries of a regular tetrahedron. $\square$

Note that the above proof shows, in particular, that the line through the midpoints of a pair of opposite edges of a regular tetrahedron is perpendicular to these edges. This is not immediately obvious and other proofs using high school geometry techniques are much more complicated. Students can observe this fact by putting a wire through a paper model of a regular tetrahedron so that it enters and exits the model at the midpoints of a pair of opposite edges; rotating the model $180°$ around the wire brings it onto itself. Now, a regular hexagon also has twelve rotational symmetries (if we do not distinguish between the two sides of the hexagon and allow three dimensional rotations which flip the hexagon over) and the students can be asked to reflect on whether the tetrahedron and hexagon have the same rotational symmetry.

In this spirit, several basic notions of group theory can be explained in a very concrete way and observed by students as they make and examine the models. Here are a few examples of how a notion from group theory can be introduced by considering symmetries of the models students built:

- *Group*: a collection of all symmetries of a model
- *Subgroup*: a collection of all symmetries that fix or preserve a vertex, an edge or a face, or a coloring of vertices and/or edges and/or faces, or preserve another polyhedron inscribed in the first one. Consider, for instance, a tetrahedron inscribed in a cube (there are actually two such tetrahedra, see Figure 2a; choose one of them). We can consider all symmetries of the cube that move the tetrahedron onto itself. They form a subgroup of the group of all symmetries of the cube.
- *Homomorphism of groups*: Let us consider the effect of a symmetry of a cube on the two tetrahedra inscribed in it. The symmetry of the cube will either interchange the tetrahedra or leave them in their



(a) Two tetrahedra inside a cube (b) Cube in a dodecahedron (c) 5 cubes in a dodecahedron

Figure 2: A few Platonic relationships[1]

places. Thus, to each symmetry of the cube we can associate a permutation of the two tetrahedra, and this gives us a homomorphism from the symmetry group of the cube to the symmetric group on two letters S_2.

- *Isomorphism of groups*: There are five cubes inscribed in a dodecahedron (Figures 2b and 2c; to visualize this, students could draw the edges of the cubes on the faces of the dodecahedron, using color to distinguish between different cubes, or build wire or Zometool models). Any symmetry of the dodecahedron permutes the five cubes. This gives a homomorphism from the symmetry group of a dodecahedron to the symmetric group on five letters S_5. Restricting this homomorphism to the subgroup of rotational symmetries yields an isomorphism of the group of rotational symmetries of a dodecahedron with the alternating group A_5 of even permutations. On the other hand, while both the full symmetry group of a dodecahedron and the group S_5 contain 120 elements, the above homomorphism from the former to the latter is not an isomorphism. Having students work through the details using the models accomplishes, at students' first encounter with groups, much more than having them check that all axioms are satisfied for one group or another.

We made several other models, including some Archimedean and Catalan solids, a few stellations and compounds, exploring the rich geometry of each model. For a final project, students worked in groups to build larger models. These included a model of one of the Stewart toroids, pictured below. Several of Stewart toroids make very attractive models, and, being of genus greater than zero, allow for a discussion of topology and, in particular, generalizations of Euler's formula. Step by step account of the making of the model in Figure 3 can be found in [2].

Figure 3: Rotunda drilled icosidodecahedron, a model by L. Goel and E. Traister

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