Teaching Combinatorics with "Poly-Universe"

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Abstract

For teaching combinatoric ideas to primary and secondary school children, we summarize our research group's recent work, informed by Tamás Varga's Complex Mathematics Education Experiment and using a recently created, visual manipulative set called Poly-Universe.

Introduction

In 2014 the Hungarian Academy of Sciences offered an open grant opportunity for the first time to support interdisciplinary research and its practical applications that are either new or build on the Hungarian didactics traditions. Our research group received a grant for our proposal "Complex Mathematics Education in the 21st Century – Improving combinatorial thinking based on the newest research results". Researchers from three Hungarian universities worked out the key ideas of the project, several primary and secondary schools participated in the pilot teaching.

Our goal was to apply Tamás Varga's ideas [2] based on the Complex Mathematics Education Experiment of the 1960s and 70s in Hungary, to update them and to build in the results of the mathematical didactics research of the past three decades. Our research was limited to a narrow area of mathematics, combinatorics education.

One reason for our choice was that most combinatorial problems cannot be solved mechanically, they require critical thinking, strategic planning, thereby improving mathematical performance and they are perfect for the concentration within the subject and between subjects.

The Experience Workshop, which implements a lot of ideas going back to Tamas Varga, participated in the project as a cooperating partner. The Experience Workshop's creative, workshop-type lessons and extracurricular activities based on practical activities and focusing on cooperative learning and experiments draw the attention to the cultural embeddedness of mathematical thinking [5].

The Importance of Using Manipulatives in Mathematics Education

The mathematics learning process is compromised if there is insufficient time for illustrations, for thorough, unhurried explorations and studying of the connections. The learning process takes place all too often at the symbolic level. The use of well-chosen tools strengthens the involvement of the material level of thinking. According to Bruner's (Bruner [1]) theory the flexibility of the transition between representational planes improves creative thinking.

An important learning theory achievement of Tamas Varga's complex mathematics education experiment was the achievement of acquisition of knowledge based on personal experiences. He wanted to offer all children the possibility of gaining personal experiences in a sufficiently wide range – organizing relevant tangible, manual and intellectual activities for them – so that they can gain knowledge from generalization and abstraction of their experience. This kind of learning demands more tools, a

significantly larger amount and more difficult organization, other kind of teaching management, and other testing than the traditional, passive learning.

The Lesson Using the Modeling Set Poly-Universe and Experiences

The pilot took place in Grades 7 and 8 in primary schools and in Grades 10 and 11 in secondary schools in a total of 8-10 hours. In the primary schools children used color bars and logic sets. For the three secondary groups, in addition to the above, we wanted to use a newly developed tool, connected to the mathematics – art education research of the past years. We thought a lot about the manipulative to be used. In the end we chose the Poly-Universe set which includes the benefits of the worksheets that can be found in the workbook, the parallel development of the combinatorial and geometric skills and a professional design, so we could buy it at an affordable price for the classes in our research.

The Poly-Universe set is made up of three basic shapes: triangle, square, and (almost) circle. The colors are: red, yellow, green, and blue. The shapes are colored in every possible way using these four colors. We purchased the boxes containing the "circle" shaped forms. The diameter of the “circles” is 9 cm, the thickness is 0.5 cm, having the same color on both sides. There are three semicircles attached to the boundary of the circle in directions making 120^{o} angles with one another, the largest has radius that is half of the original circle, for the medium it is one fourth, and for the smallest it is one eighth. The diameters of the three semicircles cut off three segments from the original circle so the basic form is not exactly a circle. So this shape is bordered by 3 arcs and 3 line segments, so they can stand on their line segment parts in a stable way, as kids tried it. The forms having the same shape and colored in all possible ways are packed in a box (Figure 1).

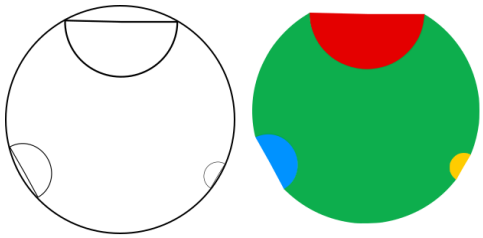

Figure 1: *The “circle” element of the set Poly-Universe.*

The first question on our combinatorial worksheet was that in this way how many elements are there in a box. Here soon emerged as an interpretation question whether all four colors should be used or not, and whether all the different colorings were made only in one copy. After answering these it was easy to obtain the correct answer, 4! = 24.

Each student got a worksheet, where, in addition to the problem, there was also a drawing of a possible solution. Teachers got the solutions and suggestions on teaching. In the classroom children worked in pairs with the set. They did the problems in order, after some time for thinking the problems were discussed, the blackboard and students' notebooks were not used.

According to the teachers' reports students quickly became fond of the Poly-Universe set. Probably no one knew it from earlier. They enjoyed moving the parts around. (Many wandered away from their task and used the elements of the set to carry out other "experiments" such as put the elements on their edge in a line and tore them down as a chain reaction or built other structures from them. The teacher tried to move them back to their task – not always successfully –, but promised them that later there will be a time when they can put together anything they like from the set.)

A Few Questions from the Worksheet

1. We select pieces of the same base color from the Poly-Universe. We put them around in a circle as on the left side of Figure 2. Sizes and colors should be the same where the semicircles join.
    a. How many circles can we make this way?
    b. Are there figures that are symmetric about a line among these objects? Are there any with rotational symmetry? How would you answer these two questions, if you would ignore the colors, and would consider only sizes?

The student worksheet contained a solution on Figure 2. Based on this solution students found two similar solutions easily by changing the colors. The touching circles inside can have two colors alternating and can be of three different sizes. In one class students found the other solutions using the teacher's guiding questions only. In these the touching circles inside have two sizes and three colors alternating. Here there are three solutions too, if the rotated and reflected solutions are not considered different. The Teachers' Guide contains these six drawings. The other group found these possibilities on their own. Soon students raised the question: Which cases should be considered different. They decided that the rotated ones are not different but the reflected ones are. So the final answer is 12. If the base color can also be changed, then the answer will be 4 times larger. After finding the 12 cases they discussed symmetry.

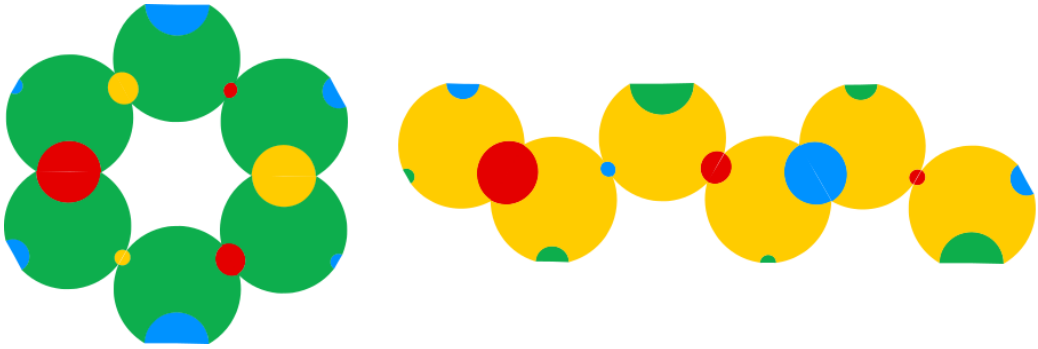

Figure 2: *Example for a closed and one for an open complete joining of circles using Poly-Universe set.*

2. On the right side of Figure 2 we made a "wavy" chain from 6 figures of the same base color so that semicircles of the same size and color join together.
    a. How many such chains of length six can be made?
    b. Can you see a connection with the previous problem?

The Teachers' Guide contained the following: This problem can be solved easily using the previous problem. Every circle solution can be changed to a wave using 6 different starting elements, the two elements at the ends can be connected in two different ways (can be switched over keeping the connections). Therefore the number of solutions is 12 * 6 * 2 * 2 = 288.

Kids started the solution differently, but several groups were on a right path, two groups got the correct final answer using multiplications. Everyone agreed that the first element can be 6 kinds in 3 different positions and these 6*3 possibilities can be continued in 2 ways. The other 2 terms were not discussed precisely, many found that in the next step sometimes we have 2 possibilities, sometimes only1. They examined together how a chain can be made from a circle. They found the solution with the teachers' help.

3. Solve the complete joining of circles problem: using all elements of a box make six circles joining together in 2-2 elements.

Students started this problem together, but only 10 minutes were left from the class period. There was not enough room on the tables for the complete set, so couples worked on the floor. Most of them were able to join 2, perhaps 3 circles, and everyone felt that it is easy to get stuck after some steps. Their teacher promised to get back to this problem later, probably in a study group.

Summary and Conclusion

During solving the problems lively discussions developed, some pairs found the correct answers on their own, others needed more or less help. There were some groups where the correct solutions were developed by collecting the answers from the couples.

We and the teachers in the project think based on the experiences of the lessons that using manipulatives is useful for secondary students too. The Poly-Universe set provides an opportunity for generalization in some tasks, but it is more common that rules cannot be applied automatically, tasks cannot be solved mechanically, which improves independent thinking and creativity. The possibility of linking combinatorics and geometry is also important. We can formulate a large number of tasks for the children about complete or partial circle fit (joining circles of different sizes), creating closed or open figures, giving different conditions for the colors. It is worth to use the device in the schools, developing additional tasks.

Figure 3: *The complete joining of circles problem.*

References

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[5] http://www.elmenymuhely.hu/?lang=en