Flatscape of Measure Polytopes

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Abstract

The Flatscape is a two-dimensional systematic representation of a higher dimensional regular polytope. In using the Genesa numbering system to identify the value of each element of the regular polytope it allows one to observe number patterns; by adding color, tile patterns are displayed.

Measure Polytope

In Section 7.2 of *Regular Polytopes* Donald Coxeter defines the measure polytope with

$$7.25 \qquad N_k = 2^{n-k}\binom{n}{k}$$

Where N_k is the number of k dimensional cells in a n-dimensional measure polytope. Coxeter then goes on to define the *orthotope* as a parallelope that has vectors that are mutually perpendicular at each vertex and says:

> If the n perpendicular vectors all have the same magnitude, the orthotope is a *hyper-cube* or *measure polytope,* γ_n**, …** The name "measure polytope" is suggested by the use of the hyper-cube of edge 1 as the unit of *content* (e.g., the square as the unit of area, and the cube as the unit of volume).

Using the formula described by Coxeter in section 7.25, we can, by induction, show that the sum of all the elements (vertices, edges, faces, cells, …) of any measure polytope, γ_n, is 3^n see Coxeter [1].

Assigning Systematic Numbers

In the 1950's the Genesa numbering was developed by Derald G. Langham to understand and classify crop genetics. This concept has been adapted as a method for numbering the elements of a polytope.

A Flatscape is a two-dimensional mapping that orders and relates all the elements of any measure polytope in any dimension. Let each element of a measure polytope be represented by a square tile. Let all of these tiles be of equal size, regardless of their dimension as the element N_k.

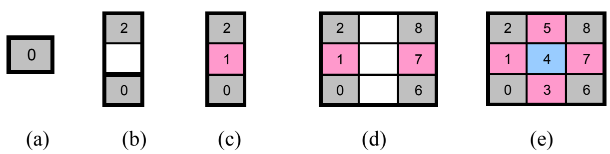
As the dimension increases, the total number of elements increases in powers of three, therefore it is natural to arrange these unit tiles in ever increasing 3 x 3 larger square mapping. That is, for any n-dimensional measure polytope, the total unit tiles 3^n are arranged in ever expanding square mappings, starting with the unit tile and proceeding in exponential steps (i.e. $3^0, 3^1, 3^2, \ldots, 3^n$).

Two-dimensional Representation of a *N*-dimensional Measure Polytope

This representation will be built in the xy plane. Let this xy plane be covered with finitely many unit square tiles. Each unit tile will represent an element of the measure polytope. The elements will be systematically numbered using base 3. This numbering will arrange the polar elements, which are defined as being additively symmetric about the center cell, Π_n. The sum of polar opposite cells will be twice the value of the center tile of the Flatscape of a measure polytope. The central tile of a Flatscape is called Π_n, and represents the measure polytope itself. Additionally for each Π_{n-k} cell, the polar elements will sum to twice the value of the center element of that particular Π_{n-k}.

Let a unit square represent a point, Π_0, to which is assigned a value of zero, see Figure 1a. To create the representation of a unit vector Π_1, replicate this zero square and translate this replicate two-steps in the y direction, see Figure 1b.

There now exists a row of three unit squares in the y direction. The two squares (the original square and its replicate) represent the end points, the vertices of the line. The unit square that occupies the space between the two end squares represents the higher dimensional space that is created in the process of replication and translation. In this case, the square in the middle represents the line segment between two end points in Π_1. See Figure 1c.



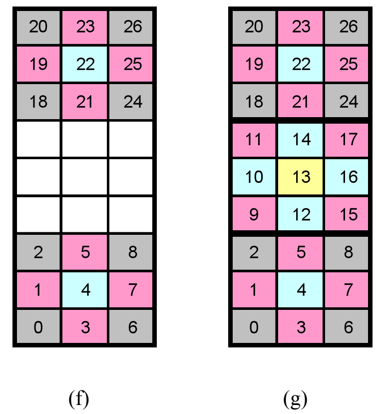


Figure 1: *Genesa numbering of a point element becoming a Flatscape cube.*

In order to assign systematic numbers to each of the elements we will take advantage of the triadic aspect of this replication and translation. That is, one element becomes two by replication and a third element is defined in the translation. That is, there is a space created between the original and its replicate.

The original square is numbered “0” and the replicated square, two steps in the y direction, is numbered “2” (2×3^0). The square that is between 0 and 2 represents the next higher dimension of the point, i.e. line. This square is numbered “1” (1×3^0).

To create the representation of the unit square (Π_2), replicate the three squares (Π_1) and translate the replicate two steps in the x direction. In moving the II_1 in the x direction, the values for the replicated three squares are increased by 3^1. Since the replicated squares have translated two steps, the net increase is two times 3^1, i.e. the replication of 0 becomes 6, the replication of 1 becomes 7, and the replication of 2 becomes 8, see figure 1d.

We now have a 3 x 3 square, made up of nine unit squares, six of which have been numbered. The three unit squares between the two II_1 are the elements in a higher dimension created by the replication and translation.

By replicating and translating the original space in a direction perpendicular to the original direction, we have “traced” out the elements of the next higher dimension.

In this case, the vertex elements have become edges and the edge element has become a square (II_2). Since these three unit squares have moved one step from the original three unit squares, in the 3^1 direction, the values are 3 and 5 for the N_1 elements and 4 for the N_2 element, see figure 1e.

To construct the representation for the elements of a cube one needs to build upon the 3 x 3 square. The next step is to replicate these nine unit squares. Translate this replicate in the y direction in two steps, each step being the size of the 3 x 3 square. Since this translation represents a stretching to the next higher dimension, the numerical effect of this translation increases the value of the replicated squares by 3^2. Since the replicated square has been moved two steps; the values of the original unit squares are increased by 2×3^2 (18), see figure 1f.

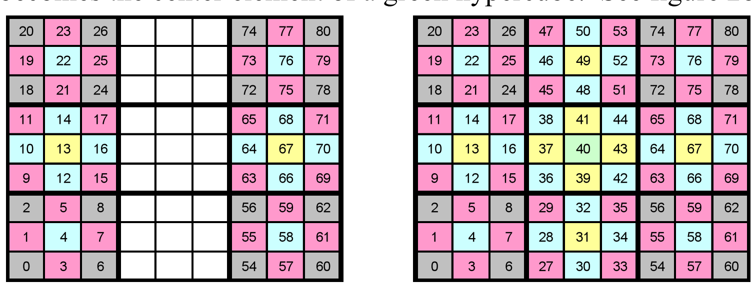
The 3 x 3 square that is created between original 3 x 3 square and its replicate has elements that are exactly one dimension higher than the elements of the original square. That is, the four vertices (N_0) become edge elements (N_1), the four edge elements become face elements (N_2), and the face element in the original square becomes a N_3 element. In total, there are 27 elements; 8 vertices (N_0), 12 edges (N_1), 6 faces (N_2), and 1 cell (N_3), see figure 1g.

Flatscape: Systematic Color Scheme for the Elements

In order to create a visual distinction of the various elements, one can assign each type of element a unique color. In these figures the N_0 elements are light gray, the N_1 elements are light red, the N_2 elements are light blue, N_3 elements are light yellow, and light green element will represent the hypercube (N_4) element. See figure 2a.

By adding a new color every time we move to a higher dimension, we can get a visual sense of the number of elements that exists for each of the measure polytopes. The hypercube (3^4) has 81 elements: 16 vertices (N_0), 32 edges (N_1), 24 faces (N_2), 8 cubes (N_3), and 1 hypercube (N_4). Applying the same method of replication and translation enables a systematic representation of the elements as 81 unit squares arranged in nine 3 X 3 squares. The translation of the replicated cube in the x direction is a stretching to a higher dimension, which in turn increase the values of the elements of the original cube by 54 and 27. The middle section created by the translation of the original cube has elements that are one-dimension higher then the elements of the original cube.

The gray vertices are now red edges, the red edges are blue faces, the blue faces are yellow cubes, and the yellow cube becomes the center element of a green hypercube. See figure 2b.



(a) (b)

Figure 2: *Flatscape hypercube.*

Opposite Pairs of Elements

Polar opposite elements always add to twice the sum of the value of their center element. This is true for all dimensions and for all elements of each measure polytope, see Figure 3.

For the hypercube the 16 grey vertices, the 32 red edges, the 24 blue faces, and the 8 yellow cubes taken in opposite pairs, total to 80, which is twice the value of the center element 40.

The elements of the hypercube also display polar numeric balance. For each of the eight cubes represented in the hypercube, the vertices, edges, and faces are symmetrically balanced, i.e. the sum of the polar opposite pairs total to twice the value of the center cube element.

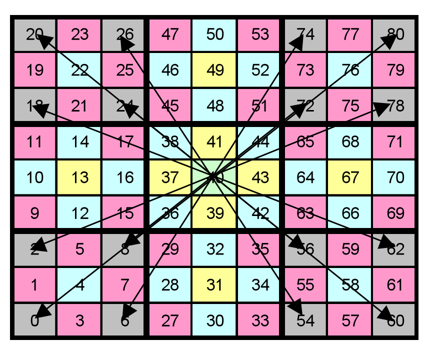


Figure 3: *Flatscape polar balance for the elements of a Hyper-cube.*

Number and Distribution of the Elements

As the dimension of the measure polytope increases the number of the elements increase. For each dimension n, there is an element N_j such that the number of any element $N_k \leq N_j$. For any N_j the number of dimensions for which N_j is the element with the maximum number of members is four. These four dimensions where N_j has the maximum number of elements are consecutive dimensions, i.e. n, n—1, n—2, n—3.

The process of replication and translation ensures that all elements increase by a factor of two plus the number of elements of one less dimension ($N_k=2N'_k + N'_{k-1}$). Since the vertices (N_0), have no element of one dimension less, the vertices simply double in number, i.e. 2,4,8,…,2^n.

For the other elements, they stay the dominant participant for four generations, sharing this dominant position with the element of one less dimension in its first generation and sharing its position of dominance with the element of one higher dimension in its last generation.

For example, in 2-dimensions, the cube (N_3), as an element does not yet exist. In 3-dimensions, the cube has one member (tile 13), in 4-Dimensions it has 8 members (13,67,39,41,31,49,37,43).

The cube (N_3) element becomes the dominant element, sharing this position with edges (N_2), in the eighth dimension, with both members having 1,792 members each. In eleventh dimension, the cube enjoys its last position of dominance sharing this position with the hypercube, each having 42,240 members. See Figure 4. Thus, as one expands the measure polytope to higher dimensions, while systematically coloring each element, one can define a Flatscape of changing colors where elements temporarily hold a position of dominance to be relinquished as higher dimensions appear.

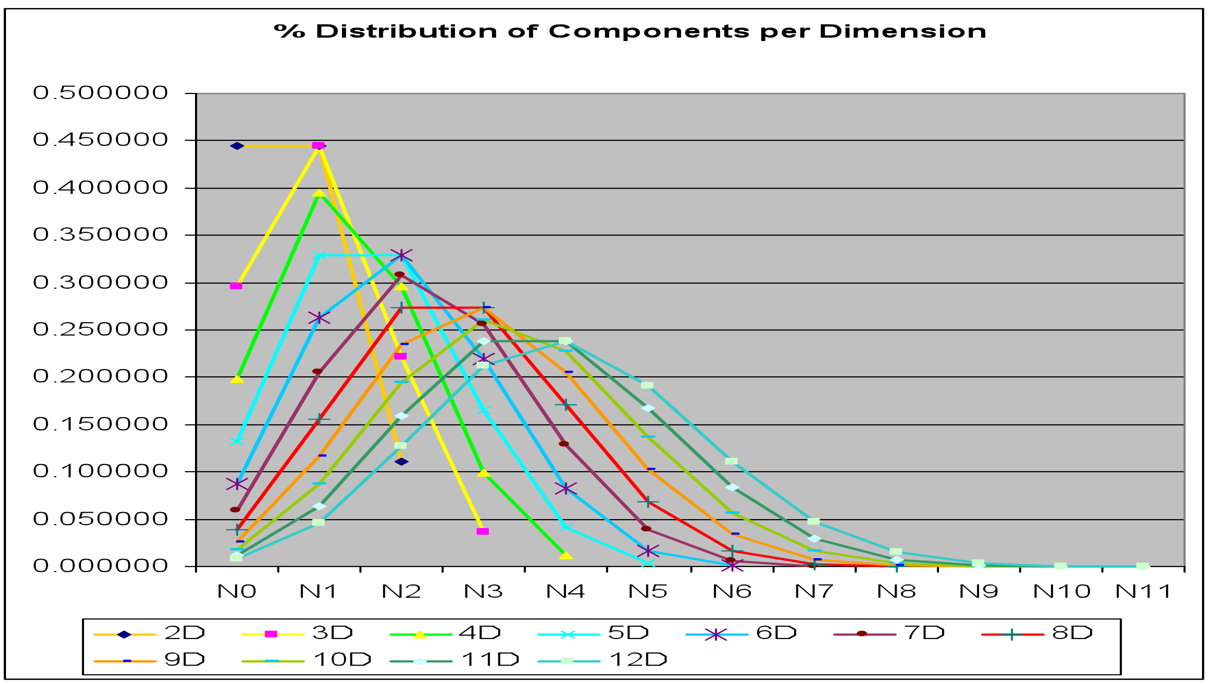


Figure 4: *Distribution of components per dimension.*

References

[1] H. S. M. Coxeter, *Regular Polytopes,* Dover Publications, 1973.